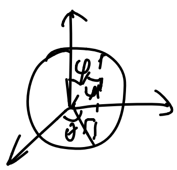
HW 13.

17.3

5. $\iint_D F \cdot d\vec{S} = \iiint_V \nabla \cdot F \, dV$



$$\begin{cases} x = r\sin\varphi\cos\theta & r: 0 \to 1 \\ y = r\sin\varphi\sin\theta & \theta: 0 \to 2\pi \\ z = r\cos\varphi & \varphi: 0 \to \pi \end{cases}$$

$\nabla \cdot F = z^2$ also since $\left|\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\right| = r^2\sin\varphi$

$$\Rightarrow \iiint_V \nabla \cdot F \, dV = \int_0^1 \int_0^{2\pi} \int_0^{\pi} (r\cos\varphi)^2 (r^2\sin\varphi) \, d\varphi \, d\theta \, dr$$

$$= \int_0^1 r^4 \, dr \int_0^{\pi} \cos^2\varphi \sin\varphi \, d\varphi \cdot 2\pi.$$

$$= \frac{1}{5} \cdot \frac{2}{3} \cdot 2\pi = \frac{4\pi}{15}.$$

11. $\begin{cases} x = r\sin\varphi\cos\theta & r: 0 \to 2 \\ y = r\sin\varphi\sin\theta & \varphi: 0 \to \frac{\pi}{2} \\ z = r\cos\varphi & \theta: 0 \to \frac{\pi}{2}. \end{cases}$ $\quad \nabla \cdot F = 3x^2 + 3z^2$

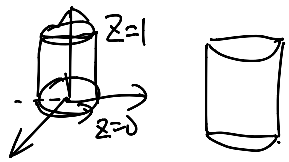
$$\iiint_V \nabla \cdot F \, dV = \int_0^2 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} 3r^2(\sin^2\varphi\cos^2\theta + \cos^2\varphi) r^2\sin\varphi \, d\varphi \, d\theta \, dr$$

$$= 3\int_0^2 r^4 \, dr \left(\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin^3\varphi \cos^2\theta \, d\theta \, d\varphi + \int_0^{\frac{\pi}{2}} \cos^2\varphi \sin\varphi \, d\varphi \cdot \frac{\pi}{2} \right)$$

$$= 3\left(\frac{r^5}{5} \Big|_0^2 \right) \left(\int_0^{\frac{\pi}{2}} \sin^3\varphi \, d\varphi \int_0^{\frac{\pi}{2}} \left(\frac{\cos 2\theta + 1}{2}\right) d\theta + \frac{\pi}{2} \cdot \frac{1}{3} \right)$$

$$= \frac{3}{5}(2^5) \cdot \left(\frac{2}{3} \cdot \frac{1}{2} \cdot \frac{\pi}{2} + \frac{\pi}{6} \right) = \frac{32}{5}\pi.$$

19.



$$4 = \iiint_W \nabla\cdot F\, dV = \iint_{\partial W} F\cdot d\vec{S}$$

$$= \iint_{\partial W_{side\text{-}curve}} F\cdot d\vec{S} + \iint_{\partial W_{up}} F\cdot d\vec{S} + \iint_{\partial W_{down}} F\cdot d\vec{S} + \iint_{\partial W_{rectangle}} F\cdot dS$$

$\iint_{\partial W_{up}} F\cdot d\vec{S}$: parametrization of up bound:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = 1 \end{cases} \quad G(r,\theta) = (r\cos\theta,\ r\sin\theta,\ 1)$$

$$\vec{N} = G_r \times G_\theta = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (0,0,r)$$

$$= \int_0^{2\pi}\int_0^1 (r^2\sin^2\theta,\ 0,0)\cdot(0,0,r)\, dr\, d\theta = 0$$

By same process we have $\iint_{\partial W_{down}} F\cdot d\vec{S} = 0$

$\iint_{\partial W_{rectangle}} F\cdot d\vec{S}$: parametrization:

$$\begin{cases} x = 0 \\ y = y \\ z = z \end{cases} \quad G(y,z) = (0,y,z)$$

$$G_y \times G_z = \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (1,0,0)$$

normal vector should be outward.

So $\vec{N} = (-1,0,0) = (-G_y \times G_z)$

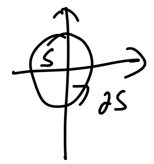
$$\Rightarrow \int_{-1}^1\int_0^1 (zy^2,0,0)\cdot(-1,0,0)\, dz\, dy$$

$$= \int_{-1}^1\int_0^1 -zy^2\, dz\, dy = (-1)\left(\int_0^1 z\, dz\right)\left(\int_{-1}^1 y^2 dy\right)$$

$$= (-1)\left(\tfrac{1}{2}\right)\left(\tfrac{2}{3}\right) = -\tfrac{1}{3}$$

So. $4 = \iint_{\partial side\text{-}curve} F\cdot d\vec{S} - \frac{1}{3} \Rightarrow \iint_{\partial side\text{-}curve} F\cdot d\vec{S} = 4 + \frac{1}{3} = \frac{13}{3}$

17.1

1. 

For the parametrization of ∂S:

$$\begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases} \quad \theta: 0 \to 2\pi$$

$$\int_0^{2\pi} (\cos\theta \sin\theta)(-\sin\theta)\, d\theta + \int_0^{2\pi} \sin\theta \cos\theta \, d\theta$$

$$= \int_0^{2\pi} (-\sin^2\theta) \cos\theta \, d\theta + 0$$

$$= 0$$

$P = xy \quad Q = y$

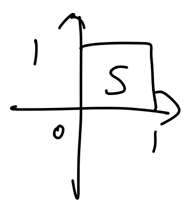
Parametrization of S:

$$\begin{cases} x = r\cos\theta & r: 0 \to 1 \\ y = r\sin\theta & \theta: 0 \to 2\pi \end{cases}$$

$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) ds = \iint_S (0 - x)\, ds$$

$$= \int_0^1 \int_0^{2\pi} (-r\cos\theta)\, r \, d\theta \, dr = 0$$

$0 = 0$ ✓

3. 

By Green's theorem we have

$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) ds$$

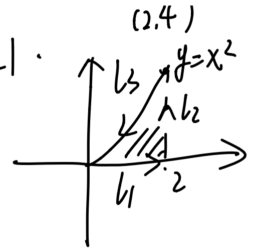
$$= \iint_S (2x - 2y)\, ds$$

$$= \int_0^1 \int_0^1 (2x - 2y)\, dx\, dy$$

$$= \int_0^1 \left(x^2 - 2yx \Big|_0^1 \right) dy$$

$$= \int_0^1 (1 - 2y)\, dy = y - y^2 \Big|_0^1 = 0$$

21.



$Q = x \quad P = 0$

$$S = \oint_{\partial D} (0)\,dx + (x)\,dy = \oint_{\partial D} x\,dy$$

$$= \int_{l_1} x\,dy + \int_{l_2} x\,dy + \int_{l_3} x\,dy$$

$\int_{l_1} x\,dy$: 0

$\int_{l_2} x\,dy$: $\int_0^4 (2)\,dy = 8$

$\int_{l_3} x\,dy$: $r(x) = (x, x^2)$, $x: 2 \to 0$

$$\int_2^0 x\,dx^2 = -\int_0^2 2x^2\,dx = -\frac{16}{3}$$

$$\Rightarrow S = 0 + 8 - \frac{16}{3} = \frac{8}{3}$$

29. $\oint_{C_1} F\cdot d\vec{r} - \oint_{C_2} F\cdot d\vec{r} - \oint_{C_3} F\cdot d\vec{r}$

$$= \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dS = 9\cdot(\pi 5^2 - 2\pi(1))$$

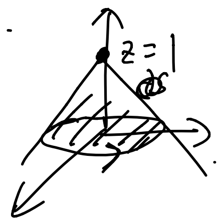
$$= 9(23\pi)$$

$$\Rightarrow \oint_{C_1} F\cdot d\vec{r} = \oint_{C_2} F\cdot d\vec{r} + \oint_{C_3} F\cdot d\vec{r} + 9(23\pi)$$

$$= 7\pi + 207\pi = 214\pi$$

17.2.

Stokes: $\oint_{\partial S} F \cdot d\vec{r} = \iint_S (\text{curl} F)\, d\vec{S}$

1. 

$\oint_{\partial S} F \cdot d\vec{r}$

parametrization: $\begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases}$

$\gamma(\theta) = (\cos\theta, \sin\theta, 0)$ $\quad (0 - e^{z^3} \cdot 3z^2) i - (-\sin x z\, z - e^{z^2} 2yz) j + (1 - (-1)$

$dr = (-\sin\theta, \cos\theta, 0)\, d\theta$

$$\oint_{\partial S} F \cdot d\vec{r} = \int_0^{2\pi} \langle 2\cos\theta\sin\theta,\ \cos\theta,\ \sin\theta \rangle \cdot \langle -\sin\theta, \cos\theta, 0 \rangle\, d\theta$$

$$= \int_0^{2\pi} (-2\sin^2\theta\cos\theta + \cos^2\theta)\, d\theta$$

$$= \int_0^{2\pi} \cos^2\theta\, d\theta = \int_0^{2\pi} \left(\frac{\cos 2\theta + 1}{2}\right) d\theta = \pi$$

$$\text{Curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) i - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) j + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) k$$

$$= (1 - 2x) k \Rightarrow (0, 0, 1-2x)$$

parametrization: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = 1 - r^2 \end{cases}$ $\quad r: 0 \to 1$, $\theta: 0 \to 2\pi$.

$\Rightarrow \quad G(r,\theta) = (r\cos\theta,\ r\sin\theta,\ 1-r^2)$

$$G_r \times G_\theta = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = 2r^2\cos\theta\, i - (-2r^2\sin\theta)j + rk$$

$$\Rightarrow (2r^2\cos\theta,\ 2r^2\sin\theta,\ r)$$

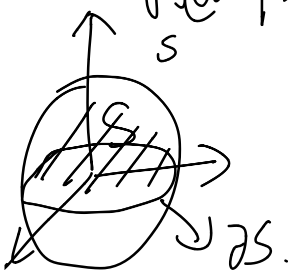
$$\int_0^{2\pi}\int_0^1 (0, 0, 1-2r\cos\theta)\cdot(2r^2\cos\theta, 2r^2\sin\theta, r)\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 (r - 2r^2\cos\theta)\, dr\, d\theta$$

$$= \frac{1}{2} \times 2\pi = \pi.$$

$\pi = \pi$ ✓

5. $\iint_S (\text{curl}\,\vec{F}) \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{r}$



The parametrization of the ∂S

$$\partial S: \begin{cases} x = \cos\theta \\ y = \sin\theta \\ z = 0 \end{cases} \qquad \theta: 0 \to 2\pi.$$

$$r(\theta) = (\cos\theta, \sin\theta, 0)$$

$$dr = (-\sin\theta, \cos\theta, 0)\, d\theta$$

$$\vec{F} = \langle 1 - \sin\theta, 1 + \cos\theta, 1 \rangle$$

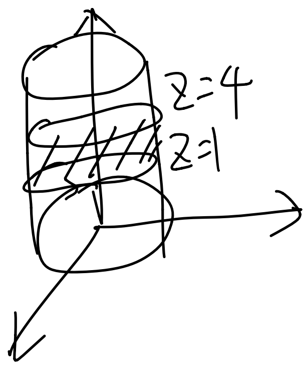
$$\int_0^{2\pi} \langle 1 - \sin\theta, 1 + \cos\theta, 1 \rangle \cdot \langle -\sin\theta, \cos\theta, 0 \rangle \, d\theta$$

$$= \int_0^{2\pi} (-\sin\theta + \sin^2\theta + \cos\theta + \cos^2\theta)\, d\theta$$

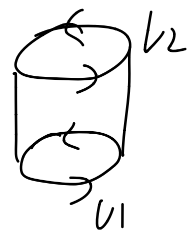
$$= 2\pi.$$

$$\text{curl}\,\vec{F} = \langle -3z^2 e^{z^3}, 2z e^{z^2} + z\sin(xz), 2 \rangle$$

9.



$$\iint_S (\text{curl}\,\vec{F}) \cdot d\vec{S}$$



$$= \oint_{\partial S} \vec{F} \cdot d\vec{r}$$

$$= \oint_{l_1} \vec{F} \cdot d\vec{r} + \oint_{l_2} \vec{F} \cdot d\vec{r}$$

$$l_1: \begin{cases} x = \cos\theta \\ y = \sin\theta \\ z = 1 \end{cases} \quad \theta: 0 \to 2\pi \quad \gamma(\theta) = (\cos\theta, \sin\theta, 1) \quad dr = (-\sin\theta, \cos\theta, 0)\, d\theta$$

$$\vec{F} = \langle \sin\theta,\ \cos\theta,\ \cos\theta \sin\theta \rangle$$

$$\int_0^{2\pi} \langle \sin\theta, \cos\theta, \cos\theta \sin\theta \rangle \cdot \langle -\sin\theta, \cos\theta, 0 \rangle \quad d\theta$$

$$= \int_0^{2\pi} \cos 2\theta \, d\theta = 0$$

C_2: $\begin{cases} x = \cos\theta \\ y = \sin\theta \\ z = 4 \end{cases}$

$\vec{F} = \langle 4\sin\theta, -4\cos\theta, 4^3 \rangle$

$= \langle 4\sin\theta, -4\cos\theta, 64 \rangle$

$r(\theta) = \langle \cos\theta, \sin\theta, 4 \rangle$

$$\int_0^{2\pi} \langle 4\sin\theta, -4\cos\theta, 64 \rangle \cdot \langle -\sin\theta, \cos\theta, 0 \rangle \, d\theta$$

$= 0$

So: $\iint_S (\operatorname{curl} \vec{F}) \cdot d\vec{S} = 0$,