HW 3

Section 14.4

1. $L(x,y) = f(-1,2) + f_x(-1,2)(x+1) + f_y(-1,2)(y-2)$

$f(-1,2) = 2(-1)^2 - 4(-1)2^2 = 2+16 = 18$

$f_x(-1,2) = 4x - 4y^2 \big|_{(-1,2)} = -4 - 4(2)^2 = -20$

$f_y(-1,2) = -8xy \big|_{(-1,2)} = 16$

$\Rightarrow L(x,y) = 18 + (-20)(x+1) + 16(y-2)$

$= -20x + 16y - 34$

11. Assume the point is (x,y,z), then the x-direction vector at the point is $(1,0,z_x)$

$\Rightarrow n \cdot (1,0,z_x) = 0$

$\Rightarrow (3,2,2) \cdot (1,0,6x) = 0 \Rightarrow 3 + 12x = 0 \Rightarrow x = \frac{-1}{4}$

also the y-direction vector at the point is

$(0,1,z_y)$

$\Rightarrow n \cdot (0,1,z_y) = 0$

$\Rightarrow (3,2,2) \cdot (0,1,-8y) = 0 \Rightarrow 2 - 16y = 0 \Rightarrow y = \frac{1}{8}$

$\Rightarrow$ The point is $(-\frac{1}{4}, \frac{1}{8}, \frac{1}{8})$

12. Assume the point is (x, y, z), then the x-direction vector $(1, 0, z_x)$ should be vertical to $(2, 7, 2)$

$\Rightarrow (1, 0, y^3) \cdot (2, 7, 2) = 0$

$\Rightarrow 2 + 2y^3 = 0 \Rightarrow y = -1$

also the y-direction vector $(0, 1, z_y)$ should be vertical to $(2, 7, 2)$ too

$\Rightarrow (0, 1, 3xy^2 - 8y^{-2}) \cdot (2, 7, 2) = 0$

$\Rightarrow 7 + 2(3xy^2 - 8y^{-2}) = 0$

$\Rightarrow 7 + 2(3x(-1)^2 - 8(-1)^{-2}) = 0$

$\Rightarrow x = \frac{3}{2}$

$\Rightarrow$ The point is $(\frac{3}{2}, -1, \frac{-19}{2})$

44. The tangent plane at $(0,2)$ is

$$L(x,y) = f(0,2) + f_x(0,2)(x-0) + f_y(0,2)(y-2)$$
$$= 4x$$

$$\Rightarrow \quad e(x,y) = f(x,y) - L(x,y)$$
$$= xy^2 - 4x$$

Then in order to prove $f(x,y)$ is differentiable at $(0,2)$. We need to prove

$$\lim_{(x,y)\to(0,2)} \frac{e(x,y)}{\sqrt{x^2+(y-2)^2}} = 0.$$

Let $\begin{cases} x = r\cos\theta \\ y = 2 + r\sin\theta \end{cases}$ then we have

$$\lim_{(r,\theta)\to(0,??)} \frac{r\cos\theta\,(2+r\sin\theta)^2 - 4r\cos\theta}{\sqrt{(r\cos\theta)^2 + (r\sin\theta)^2}}$$

$$= \lim_{(r,\theta)\to(0,??)} \frac{r\cos\theta\,(4+4r\sin\theta+r^2\sin^2\theta) - 4r\cos\theta}{r}$$

$$= \lim_{(r,\theta)\to(0,??)} \frac{4r^2\cos\theta\sin\theta + r^3\cos\theta\sin^2\theta}{r}$$

$$= \lim_{(r,\theta)\to(0,??)} 4r\cos\theta\sin\theta + r^2\cos\theta\sin^2\theta$$

Since $\sin\theta \in [-1,1]$, $\cos\theta \in [-1,1]$

then $\sin\theta\cos\theta \in [-1,1]$

$$-4r-r^2 \leq 4r\cos\theta\sin\theta + r^2\cos\theta\sin^2\theta \leq 4r+r^2$$

$$\lim_{r\to 0} 4r-r^2 = 0, \quad \lim_{r\to 0} 4r+r^2 = 0$$

By squeeze theorem, we have

$$\lim_{(r,\theta)\to(0,??)} 4r\cos\theta\sin\theta + r^2\cos\theta\sin^2\theta = 0.$$

19. $f(x,y) \approx f(0,0) + f_x(0,0)(x) + f_y(0,0)(y)$

$f(0,0) = e^{0^2+0} = 1$

$f_x(0,0) = e^{x^2+y}(2x)\big|_{(0,0)} = 0$

$f_y(0,0) = e^{x^2+y}\big|_{(0,0)} = 1$

$\Rightarrow f(x,y) \approx 1 + (y) = 1+y$

$\Rightarrow f(0.01, -0.02) \approx 1-0.02 = 0.98$

By calculator we have

$f(0.01, -0.02) = e^{0.01^2-0.02} = 0.9803$

23. $f(x,y) \approx f(2,4) + f_x(2,4)(x-2) + f_y(2,4)(y-4)$

$\Rightarrow f(2.1, 3.8) \approx 5 + 0.3(2.1-2) + (-0.2)\cdot(3.8-4)$

$= 5 + 0.3(0.1) + (-0.2)(-0.2)$

$= 5.07$

31. suppose (x, y, z) is on the plane

then $(x, y, z) - (-2, 3, 4) = (x+2, y-3, z-4)$

is othogonal to the vector $(4, 2, 1)$

$\Rightarrow (x+2, y-3, z-4) \cdot (4, 2, 1) = 0$

$\Rightarrow 4(x+2) + 2(y-3) + (z-4) = 0$

$\Rightarrow z = -4(x+2) - 2(y-3) + 4$

$\Rightarrow f(-2.1, 3.1) \approx -4(-2.1+2) - 2(3.1-3) + 4$

$= 0.4 - 0.2 + 4 = 4.2$

47. (a). Let $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ then $g(x, y)$ changes

into a function of $g(r, \theta)$

$$g(r,\theta) = \begin{cases} 2r\cos\theta\sin\theta\,(\sin\theta+\cos\theta) & r \neq 0 \\ 0 & r = 0 \end{cases}$$

$$\lim_{r\to 0} g(r,\theta) = \lim_{r\to 0} 2r\cos\theta\sin\theta\,(\sin\theta+\cos\theta)$$

$$-4r \leq 2r\cos\theta\sin\theta\,(\sin\theta+\cos\theta) \leq 4r$$

$$\lim_{r\to 0} 4r = 0, \quad \lim_{r\to 0}(-4r) = 0$$

By squeeze theorem, we have

$$\lim_{r\to 0} 2r\cos\theta\sin\theta\,(\sin\theta+\cos\theta) = 0$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} g(x,y) = 0 = g(0,0)$$

$\Rightarrow$ g is continuous at $(0,0)$.

(b). $$g_x(0,0) = \lim_{\Delta x\to 0} \frac{g(\Delta x, 0) - g(0,0)}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{2(\Delta x)(0)(\Delta x+0)}{(\Delta x)^2(\Delta x)}$$

$$= \lim_{\Delta x\to 0} \frac{0}{(\Delta x)^3} = 0$$

$$g_y(0,0) = \lim_{\Delta y \to 0} \frac{g(0,\Delta y) - g(0,0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{2(0)(\Delta y)^2}{(\Delta y)^3} = 0$$

(c). $L(x,y) = g(0,0) + g_x(0,0)(x-0) + g_y(0,0)(y-0)$

$= 0 + 0(x) + 0(y) = 0$

(d) If $g(x,y)$ were differentiable at $(0,0)$ then by definition we have.

$$\lim_{(x,y)\to(0,0)} \frac{g(x,y) - L(x,y)}{\sqrt{x^2+y^2}} = 0$$

Since $L(x,y)=0$

$$\Longrightarrow \lim_{(x,y)\to(0,0)} \frac{g(x,y)}{\sqrt{x^2+y^2}} = 0$$

change x, y into "h" here

$$\Longrightarrow \lim_{(x,y)\to(0,0)} \frac{g(h,h)}{\sqrt{2h^2}} = 0 \Rightarrow \lim_{h\to 0} \frac{g(h,h)}{h} = 0$$

However, when we plug in the function of g, we have

$$g(h,h) = \frac{2h^2(h+h)}{h^2+h^2} = 2h$$

$$\Rightarrow \lim_{h\to 0} \frac{g(h,h)}{h} = \lim_{h\to 0} \frac{2h}{h} = 2 \neq 0$$

this is a contradiction with the result when g is differentiable

so $g(x,y)$ is not differentiable at $(0,0)$.

Section 14.5

26. $D_v f(1,0) = \nabla f(2,2) \cdot \frac{v}{\|v\|}$

$$\nabla f = (e^{xy-y^2}(y),\ e^{xy-y^2}(x-2y))$$

$$\nabla f \big|_{(2,2)} = (2, -2)$$

$$D_v f(2,2) = (2,-2) \cdot \left(\frac{12}{13}, \frac{-5}{13}\right)$$
$$= \frac{34}{13}$$

31. $\vec{u} = (0,0) - (3,2) = (-3,-2)$

$$D_u f(3,2) = \nabla f_{(3,2)} \cdot \frac{\vec{u}}{\|u\|}$$

$$\nabla f_{(3,2)} = (2x, 8y) \big|_{(3,2)}$$
$$= (6, 16)$$

$$D_u f_{(3,2)} = (6,16) \cdot \left(\frac{-3}{\sqrt{13}}, \frac{-2}{\sqrt{13}}\right)$$

$$= \frac{-50}{\sqrt{13}}$$