Hw 5

Section 14.7

15. $\nabla f=0 \Rightarrow \begin{cases} ye^{-y^2}(e^{-x^2}+x(-2x)e^{-x^2})=0 \\ xe^{-x^2}(e^{-y^2}-2y^2e^{-y^2})=0 \end{cases}$

$\Rightarrow \begin{cases} y(1-2x^2)=0 \\ x(1-2y^2)=0 \end{cases} \Rightarrow \begin{cases} y=0 \Rightarrow x=0 \quad (0,0) \\ x^2=\frac{1}{2} \Rightarrow y^2=\frac{1}{2} \quad (\pm\sqrt{\frac{1}{2}}, \pm\sqrt{\frac{1}{2}}) \end{cases}$

$D=(f_{xx})(f_{yy})-(f_{xy})^2$

$= e^{-2x^2-2y^2}\left[(4x^3-6x)(4y^3-6y)xy-(1-2y^2)^2(1-2x^2)^2\right]$

at critical points: $D(0,0)<0$

$D(\sqrt{\frac{1}{2}},\sqrt{\frac{1}{2}})>0$, $D(-\sqrt{\frac{1}{2}},-\sqrt{\frac{1}{2}})>0$

$D(-\sqrt{\frac{1}{2}},\sqrt{\frac{1}{2}})>0$ $D(\sqrt{\frac{1}{2}},-\sqrt{\frac{1}{2}})>0$

$f_{xx}=(-4xy)e^{x^2-y^2}+y(1-2x^2)e^{-x^2-y^2}(-2x)$

$=(-6xy+4x^3y)e^{-x^2-y^2}=4xy(x^2-\frac{3}{2})e^{-x^2-y^2}$

Therefore, $(0,0)$ is a saddle point.

$f_{xx}(\sqrt{\frac{1}{2}},\sqrt{\frac{1}{2}})=f_{xx}(-\sqrt{\frac{1}{2}},-\sqrt{\frac{1}{2}})<0 \Rightarrow (\sqrt{\frac{1}{2}},\sqrt{\frac{1}{2}}),(-\sqrt{\frac{1}{2}},-\sqrt{\frac{1}{2}})$ is a local maximum

$f_{xx}(-\sqrt{\frac{1}{2}},\sqrt{\frac{1}{2}})=f_{xx}(\sqrt{\frac{1}{2}},-\sqrt{\frac{1}{2}})>0 \Rightarrow (\sqrt{\frac{1}{2}},-\sqrt{\frac{1}{2}}),(-\sqrt{\frac{1}{2}},\sqrt{\frac{1}{2}})$ is a local minimum

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17. $\nabla f=0$ then $(\cos(x+y)+\sin x, \cos(x+y))=0$

$\Rightarrow \begin{cases} \cos(x+y)+\sin(x)=0 \\ \cos(x+y)=0 \end{cases} \Rightarrow \begin{cases} \sin(x)=0 \\ \cos(x+y)=0 \end{cases}$

$0=\cos(x+y)=\cos(x)\cos(y)-\sin(x)\sin(y)=\cos(x)\cos y \Rightarrow \cos(y)=0$

$$D = f_{xx} f_{yy} - (f_{xy})^2 = (-\sin(x+y) + \cos x)(-\sin(x+y)) - (\sin(x+y))^2$$

$$= -\cos x \sin(x+y) = -\cos x (\sin x \cos y + \cos x \sin y) = -\cos^2 x \sin y$$

$$f_{xx} = -\sin(x+y) + \cos x$$
$$= -(\sin x \cos y + \cos x \sin y) + \cos x$$
$$= -\cos x \sin y + \cos x = \cos x (1 - \sin y)$$

at $\sin y = 1$ $D < 0$ Saddle $\Rightarrow (k\pi, 2k\pi + \frac{\pi}{2})$ $k \in \mathbb{Z}$

$\sin y = -1$ $D > 0$ $\begin{cases} \cos x = 1 & \text{local minimal} \\ \cos x = -1 & \text{local maximal} \end{cases}$

$\Rightarrow \begin{cases} (2k\pi, 2k\pi + \frac{3\pi}{2}) \ k \in \mathbb{Z} & \text{local minimal} \\ (2k\pi + \pi, 2\pi k + \frac{3\pi}{2}) \ k \in \mathbb{Z} & \text{local maximum} \end{cases}$

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23. $f(x,y) = (x+3y) e^{y-x^2}$ $\quad e^{y-x^2} + (x+3y) e^{y-x^2}(-2x)$

$\nabla f = 0 \Rightarrow \left((1 - 2x(x+3y)) e^{y-x^2}, (x+3y+3) e^{y-x^2} \right) = 0$

$\Rightarrow \begin{cases} 1 - 2x(x+3y) = 0 \\ (x+3y) + 3 = 0 \end{cases} \Rightarrow \begin{cases} x = -\frac{1}{6} \\ y = \frac{-17}{18} \end{cases}$

$$D = f_{xx} f_{yy} - (f_{xy})^2 = (-6x - 6y + 4x^3 + 12x^2 y)(6 + x + 3y) e^{2y - 2x^2}$$
$$- (-6x + 1 - 2x^2 - 6xy)^2 e^{2y-2x^2}$$

$D(-\frac{1}{6}, \frac{-17}{18}) > 0$ $\quad \Rightarrow$ It's a local minimal.

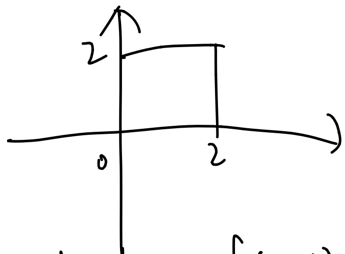
$f_{xx}(-\frac{1}{6}, \frac{-17}{18}) = (-6x - 6y + 4x^3 + 12x^2 y) e^{y-x^2} \Big|_{(-\frac{1}{6}, \frac{-17}{18})} > 0$

40. $\nabla f = 0 \quad (y-2x,\ 2y+x) = 0$

$\Rightarrow y = 2x \quad y = \frac{-x}{2} \Rightarrow x = y = 0$

$\Rightarrow (0,0)$

Boundary:



$x=0 \quad 0 \le y \le 2: \quad f(0,y) = y^2$

critical points: $(0,0)$ boundary: $(0,0)$, $(0,2)$

$y=2 \quad 0 \le x \le 2: \quad f(x,2) = 4 + 2x - x^2$

critical points: $(1,2)$ boundary: $(0,2)$, $(2,2)$

$x=2,\ 0 \le y \le 2: \quad f(2,y) = y^2 + 2y - 4$

critical points: None $\left((y^2+2y-4)' = 0 \Rightarrow 2y+2=0 \Rightarrow y=-1 \right)$

boundary: $(2,0)$, $(2,2)$

$y=0,\ 0 \le x \le 2: \quad f(x,0) = -x^2$

critical points: $(0,0)$ boundary: $(0,0)$ $(2,0)$

Combining all points we have:

$(0,0)$	0
$(1,2)$	5
$(0,2)$	4
$(2,2)$	4
$(2,0)$	-4

$\Rightarrow$ max 5 at $(1,2)$

7. $\nabla f = \lambda \nabla g$

$\Rightarrow (y, x) = \lambda (8x, 18y)$

$\Rightarrow \frac{y}{8x} = \frac{x}{18y} \Rightarrow 18y^2 = 8x^2 \Rightarrow 9y^2 = 4x^2$

$\Rightarrow x^2 = \frac{9y^2}{4}$

$4x^2 + 9y^2 = 32$

$\Rightarrow 18y^2 = 32. \Rightarrow y^2 = \frac{16}{9} \Rightarrow x^2 = 4$

we get points $(\pm 2, \pm \frac{4}{3})$

max $f(x, y) = \frac{8}{3}$ at $(2, \frac{4}{3})$ $(-2, -\frac{4}{3})$

min $f(x, y) = -\frac{8}{3}$ at $(-2, \frac{4}{3}), (2, -\frac{4}{3})$

21. $\begin{cases} \max: f(x,y) = x \\ x^2 + 6y^2 + 3xy = 40 \end{cases}$

$\nabla f = \lambda \nabla g$

$(1,0) = \lambda(2x+3y,\ 12y+3x)$

$\Rightarrow \begin{cases} 1 = \lambda(2x+3y) \\ 0 = \lambda(12y+3x) \end{cases} \Rightarrow 12y+3x=0$

$\Rightarrow \quad y = -\frac{x}{4}$

$\Rightarrow \quad x^2 + 6\left(\frac{x^2}{16}\right) + 3x\left(-\frac{x}{4}\right) = 40$

$\Rightarrow \quad \left(1 + \frac{3}{8} - \frac{3}{4}\right)x^2 = 40$

$\frac{5}{8}x^2 = 40 \Rightarrow x^2 = 64$

$\Rightarrow x = \pm 8$

then $(8,-2)$ is the points with max x.