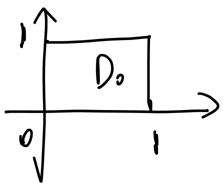
HW 9.

21. $\begin{cases} x = 5u+3v \\ y = u+4v \end{cases}$



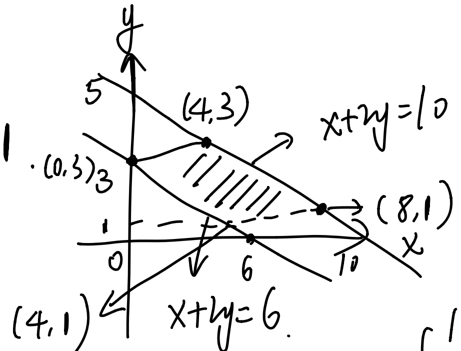
$$\iint_{D_0} (xy)\cdot\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv.$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 5 & 3 \\ 1 & 4 \end{vmatrix} = 17$$

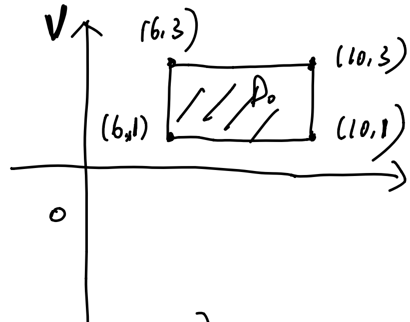
$$= \int_0^1\int_0^1 (5u+3v)(u+4v)(17)\,du\,dv.$$

$$= \frac{2329}{12}.$$

31



$\begin{cases} x = u-2v \\ y = v \end{cases}$ $\Longrightarrow$



$$\int_6^{10}\int_1^3 (u-2v+3v)\cdot\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\,du.$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = 1$$

$$= \int_6^{10}\int_1^3 (u+v)\,(1)\,dv\,du$$

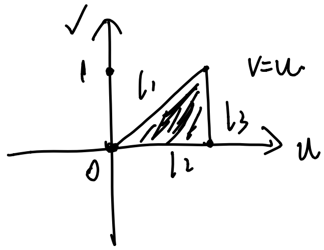
$$= \int_6^{10} \left(uv + \frac{v^2}{2}\Big|_1^3\right) du.$$

$$= \int_6^{10} \left(3u + \frac{9}{2} - u - \frac{1}{2}\right) du$$

$$= \int_6^{10} (2u+4)\,du = u^2+4u\Big|_6^{10}$$

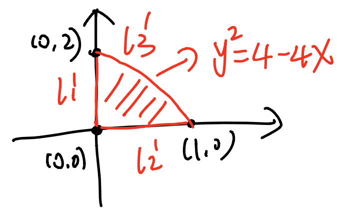
$$= 80$$

33. 1°.



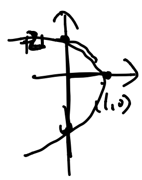
$x = u^2 - v^2$

$y = 2uv$



we consider the boundary points and line segment equation on each boundary:

l_1 : BP: $(0,0)$, $(1,1)$ $\xrightarrow{\begin{cases} x=u^2-v^2 \\ y=2uv \end{cases}}$ $\begin{cases} x=0 \\ y=0 \end{cases}$ $(0,0)$ $\begin{cases} x=1-1 \\ y=2(1)(1) \end{cases}$ $(0,2)$



line equation: $u=v$, $0 \le u \le 1$ $\Rightarrow \begin{cases} x=u^2-u^2=0 \\ y=2u^2 \quad 0 \le y \le 2 \end{cases}$

l_2: BP $(0,0)$, $(1,0)$ $\xrightarrow{\begin{cases} x=u^2-v^2 \\ y=2uv \end{cases}}$ $\begin{cases} x=0 \\ y=0 \end{cases}$ $(0,0)$ $\begin{cases} x=1^2-0^2=1 \\ y=2(1)(0)=0 \end{cases}$ $(1,0)$

line equation: $\begin{cases} v=0 \\ 0 \le u \le 1 \end{cases} \Rightarrow \begin{cases} x=u \\ y=0 \end{cases} \Rightarrow \begin{cases} 0 \le x \le 1 \\ y=0 \end{cases}$

l_3 : BP: $(1,0)$, $(1,1)$ $\Rightarrow (1,0)$ and $\begin{cases} x=1^2-1^2=0 \\ y=2(1)(1)=2 \end{cases}$ $(0,2)$

line equation:

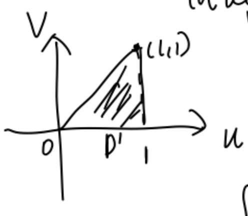
$\begin{cases} u=1 \\ 0 \le v \le 1 \end{cases} \Rightarrow \begin{cases} x=u^2-v^2 \\ y=2(1)v=2v \end{cases}$

$\Rightarrow x = 1-\left(\frac{y}{2}\right)^2$

$\Rightarrow x = 1-\frac{y^2}{4}$

$y^2 = 4-4x$

2°. Using u-v plane to evaluate the integral.



Since $\begin{cases} x = u^2 - v^2 \\ y = 2uv \end{cases}$

$$\iint_{D'} \sqrt{x^2+y^2} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA_{uv}$$

$$= \int_0^1 \int_0^u \sqrt{(u^2-v^2)^2+(2uv)^2} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dv\, du.$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2$$

$$= \int_0^1 \int_0^u \sqrt{u^4+v^4+2u^2v^2}\; 4(u^2+v^2)\, dv\, du$$

$$= \int_0^1 \int_0^u 4(u^2+v^2)^2\, dv\, du.$$

$$= 4\left(\int_0^1 \int_0^u (u^4+2u^2v^2+v^4)\, dv\, du\right)$$

$$= 4\left(\int_0^1 \left(u^4 v + 2u^2 \cdot \frac{v^3}{3} + \frac{v^5}{5} \Big|_0^u\right) du\right)$$

$$= 4\int_0^1 \left(u^5 + \frac{2u^5}{3} + \frac{u^5}{5}\right) du.$$

$$= 4\int_0^1 \left(\frac{(15+10+3)}{15} u^5\right) du.$$

$$= \frac{4\times 28}{15} \times \left(\frac{u^6}{6}\Big|_0^1\right) = \frac{4\times 28}{90} = \frac{56}{45}.$$

39 $\begin{cases} 10 \le x\overset{6}{y} \le 20 \\ 20 \le x^2 y \le 40 \end{cases}$

$\begin{cases} u = xy \\ v = x^2 y \end{cases} \Rightarrow \begin{matrix} 10 \le u \le 20 \\ 20 \le v \le 40 \end{matrix}$

$$\int_{10}^{20} \int_{20}^{40} e^u \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dv\, du$$

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{1}{\left| \frac{\partial(u,v)}{\partial(x,y)} \right|}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \begin{vmatrix} y & x \\ 2xy & x^2 \end{vmatrix}$$

$$= x^2 y - 2x^2 y = -x^2 y$$

$$= -v$$

So the original equation changes into

$$\int_{10}^{20}\int_{20}^{40} e^{u}\frac{1}{|-v|}\,dv\,du.$$

$$=\int_{10}^{20}\int_{20}^{40} e^{u}\cdot\frac{1}{v}\,dv\,du.$$

$$=\left(e^{u}\Big|_{10}^{20}\right)\left(\ln v\Big|_{20}^{40}\right)$$

$$=(e^{20}-e^{10})(\ln 40-\ln 20)=(e^{20}-e^{10})(\ln 2)$$

41. (a). from D we know

$$2\le u\le 4$$

$$0\le v\le 3$$

So the $R=[2,4]\times[0,3]$.

(b). $$\frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}=\frac{-1}{x+y}$$

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}=\begin{vmatrix} y & x \\ 1 & -1 \end{vmatrix}=-y-x$$

(c). $\iint_D (x^2-y^2)\,dx\,dy$

$\begin{cases} u = xy \\ v = x-y \end{cases}$ $= \iint_R (x^2-y^2) \cdot \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$

$= \iint_R (x^2-y^2) \frac{1}{(x+y)}\, du\,dv$

$= \iint_R (x-y)\, du\,dv$

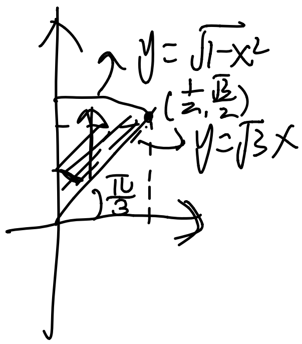
$= \int_2^4 \int_0^3 v \, dv\, du$

$= \int_2^4 \left(\frac{v^2}{2} \Big|_0^3 \right) du$

$= \frac{9}{2} \left(u \Big|_2^4 \right) = 9.$

15.4

9. $\int_0^{\frac{1}{2}} \int_{\sqrt{3}x}^{\sqrt{1-x^2}} (x)\, dy\, dx$



$\sqrt{1-x^2} = \sqrt{3}x$

$\Rightarrow 1-x^2 = 3x^2$

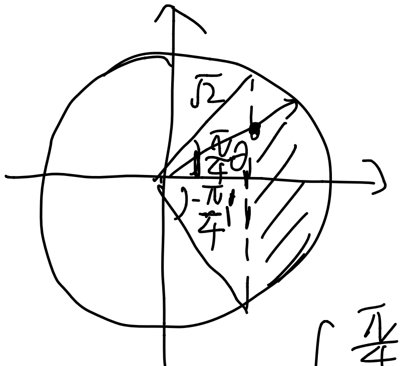
$\Rightarrow x^2 = \frac{1}{4} \Rightarrow x = \frac{1}{2}.$

$$\int_0^1 \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (r\cos\theta)\, r\, d\theta\, dr$$

$$= \int_0^1 r^2\, dr \cdot \left(\sin\theta \Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}} \right)$$

$$= \frac{1}{3}\left(1 - \frac{\sqrt{3}}{2}\right)$$

15.



$\theta : -\frac{\pi}{4} \to \frac{\pi}{4}$

$r : \frac{1}{\cos\theta} \to \sqrt{2}.$

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{\frac{1}{\cos\theta}}^{\sqrt{2}} \frac{1}{r^4}\, r \, dr\, d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{\frac{1}{\cos\theta}}^{\sqrt{2}} \frac{1}{r^3} \, dr \, d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{r^{-2}}{-2} \Big|_{\frac{1}{\cos\theta}}^{\sqrt{2}} \right) d\theta$$

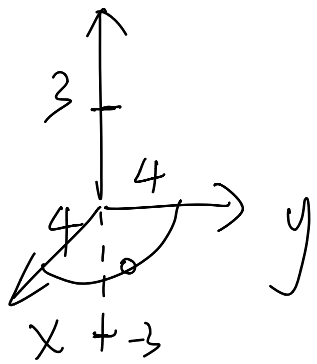
$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} \left(\cos^2\theta - \frac{1}{2} \right) d\theta.$$

$$= \frac{1}{4} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (2\cos^2\theta - 1)\, d\theta$$

$$= \frac{1}{4} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos(2\theta)\, d\theta = \frac{1}{8} \left(\sin 2\theta \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \right)$$

$$= \frac{1}{4}.$$

29.



$$\int_0^{\frac{\pi}{2}}\int_0^4\int_{-3}^{3}(r\cos\theta)(r)\,dz\,dr\,d\theta$$

$$=\left(\int_0^{\frac{\pi}{2}}\cos\theta\,d\theta\right)\left(6\int_0^4 r^2\,dr\right)$$

$$=6\cdot\left(\frac{r^3}{3}\Big|_0^4\right)$$

$$=2\cdot 4^3=128$$

37. equation: $\frac{R}{H} = \frac{\sqrt{x^2+y^2}}{z}$

$$z = \frac{H\sqrt{x^2+y^2}}{R}.$$

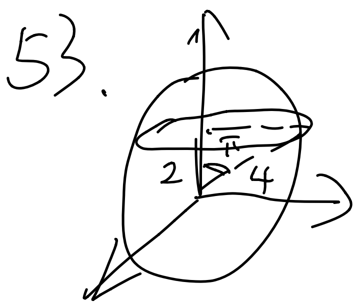
$$V = \int_0^{2\pi}\int_0^R\int_{\frac{Hr}{R}}^{H} (1)\, r \, dz\, dr\, d\theta$$

$$= 2\pi \cdot \int_0^R r\left(H - \frac{Hr}{R}\right) dr.$$

$$= 2\pi \left(H \cdot \frac{r^2}{2} - \frac{H}{R} \cdot \frac{r^3}{3} \Big|_0^R \right)$$

$$= 2\pi \left(\frac{H}{2}R^2 - \frac{H}{3}R^2 \right) = \frac{HR^2}{3}\pi.$$

53.



$\varphi: 0 \to \frac{\pi}{3}$

$\theta: 0 \to 2\pi$

$\rho: \frac{2}{\cos\varphi} \to 4$

$z = \rho\cos\varphi$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_{\frac{2}{\cos\varphi}}^{4} \rho\cos\varphi\cdot\rho^{-3}\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$=(2\pi)\int_0^{\frac{\pi}{3}}\int_{\frac{2}{\cos\varphi}}^{4}\cos\varphi\sin\varphi\,d\rho\,d\varphi$$

$$=(2\pi)\int_0^{\frac{\pi}{3}}\cos\varphi\sin\varphi\left(4-\frac{2}{\cos\varphi}\right)d\varphi$$

$\cos\varphi = u \qquad du = -\sin\varphi\,d\varphi$

$u: \frac{1}{2} \to 1$

$$=(2\pi)\int_{\frac{1}{2}}^{1} u\left(4-\frac{2}{u}\right)du.$$

$$= (2\pi) \int_{\frac{1}{2}}^{1} (4u - 2)\, du$$

$$= (2\pi) \left(2u^2 - 2u \Big|_{\frac{1}{2}}^{1} \right)$$

$$= (2\pi) \left(0 - \left(2 \cdot \frac{1}{4} - 1 \right) \right) = \pi.$$