Hw13

Section 6.1 4(b). $f(t) = t^2$

Solution:

$$\begin{aligned} L\{f(t)\} &= L\left\{t^2\right\} = \int_0^\infty e^{-st}t^2 dt = \int_0^\infty (-\frac{1}{s})t^2 de^{-st} \\ &= \int_0^\infty \frac{2}{s} te^{-st} dt = \int_0^\infty -\frac{2}{s^2} tde^{-st} = \int_0^\infty \frac{2}{s^2} te^{-st} dt = -\frac{2}{s^3} e^{-st}|_{t=0}^\infty = \frac{2}{s^3} \quad s > 0 \end{aligned}$$

6. Solution:

$$\begin{aligned} L\{f(t)\} &= L\{\cosh(bt)\} = \int_0^\infty e^{-st}\frac{e^{bt}+e^{-bt}}{2} dt = 1/2\int_0^\infty e^{-(s-b)t} + e^{-(s+b)t} dt \\ &= 1/2\frac{e^{(s-b)t}}{b-s}|_{t=0}^\infty + 1/2\frac{e^{-(s+b)t}}{-b-s}|_{t=0}^\infty \\ &= 1/2\left(\frac{1}{s-b} + \frac{1}{b+s}\right) = \frac{s}{s^2-b^2} \quad s > |b|. \end{aligned}$$

Section 6.2 3. $F(s) = \frac{2}{s^2+3s+4}$

Solution:

$$\begin{aligned} F(s) &= \frac{2}{s^2+3s+4} = \frac{2}{(s+4)(s-1)} \\ &= \frac{A}{s+4} + \frac{B}{s-1} = \frac{A(s-1)+B(s+4)}{(s+4)(s-1)} \\ &= \frac{(A+B)s - A + 4B}{(s+4)(s-1)} \end{aligned}$$

We have $\begin{cases} A+B &= 0 \\ -A+4B &= 2 \end{cases} \Longrightarrow \begin{cases} A &= 2/5 \\ B &= 2/5 \end{cases}$

Therefore $F(s) = \frac{-2/5}{s+4} + \frac{2/5}{s-1} = \frac{2}{5}\frac{1}{s-1} - \frac{2}{5}\frac{1}{s+4}$.

and

$f(t) = L^{-1}\{F(s)\} = \frac{2}{5}L^{-1}\left\{\frac{1}{s-1}\right\} - \frac{2}{5}L^{-1}\left\{\frac{1}{s+4}\right\} = \frac{2}{5}e^t - \frac{2}{5}e^{-4t}$.

14. $y^{(4)} - y = 0 \quad y(0) = 1, y'(0) = 0, y''(0) = 1, y'''(0) = 0.$

Solution:

$L\left(y^{(4)} - y\right) = L(0) = 0.$

Let $F(s) = L(y(t))$, we have $L(y^{(4)}) = s^4L(y) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = s^4F(s) - s^3 - s$

$$\begin{aligned} s^4F(s) - s^3 - s - F(s) = 0 &\implies (s^4-1)F(s) = s^3 + s \\ &\implies F(s) = \frac{s(s^2+1)}{((s^2-1)(s^2+1)} = \frac{s}{(s^2-1)}. \end{aligned}$$

$$\begin{aligned} F(s) &= \frac{s}{(s+1)(s-1)} \\ &= \frac{A}{s+1} + \frac{B}{s-1} = \frac{A(s-1)+B(s+1)}{(s+1)(s-1)} \\ &= \frac{(A+B)s - A + B}{(s+1)(s-1)} \end{aligned}$$

We have $\begin{cases} A+B &= 1 \\ -A+B &= 0 \end{cases} \implies \begin{cases} A &= 1/2 \\ B &= 1/2 \end{cases}$

Therefore $F(s) = \frac{1/2}{s+1} + \frac{1/2}{s-1} = \frac{1}{2}\frac{1}{s+1} + \frac{1}{2}\frac{1}{s-1}.$

and

$y(t) = L^{-1}\left\{F(s)\right\} = \frac{1}{2}L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{s-1}\right\} = \frac{1}{2}e^{-t} + \frac{1}{2}e^{t} = \cosh t.$

16. $y'' - 2y' + 2y = e^{-t} \quad y(0) = 0, y'(0) = 1.$

Solution:

$L\left(y'' - 2y' + 2y\right) = L(e^{-t}) = \frac{1}{s+1}.$

Let $F(s) = L(y(t))$, we have $L(y'') = s^2L(y) - sy(0) - y'(0) = s^2F(s) - 1$ and $L(y') = sL(y) - y(0) = sF(s)$.

$$\begin{aligned} s^2F(s) - 1 - 2(sF(s)) + 2F(s) = \frac{1}{s+1} &\implies (s^2 - 2s + 2)F(s) = \frac{1}{s+1} + 1 \\ &\implies F(s) = \frac{\frac{1}{s+1} + 1}{s^2 - 2s + 2}. \end{aligned}$$

$$
\begin{aligned}
F(s) &= \frac{\frac{1}{s+1}+1}{s^2-2s+2} = \frac{s+2}{(s+1)(s^2-2s+2)} \\
&= \frac{A}{s+1} + \frac{Bs+C}{s^2-2s+2} \\
&= \frac{As^2-2As+2A+Bs^2+(B+C)s+c}{(s+1)(s^2-2s+2)}
\end{aligned}
$$

We have $\begin{cases} A+B & = & 0 \\ B+C-2A & = & 1 \\ 2A+C & = & 2 \end{cases} \Longrightarrow \begin{cases} A & = & 1/5 \\ B & = & -1/5 \\ C & = & 7/5 \end{cases}$

Therefore $F(s) = \frac{1/5}{s+1} - 1/5\frac{s-1}{(s-1)^2+1} + 7/5\frac{1}{(s-1)^2+1}$.

and

$y(t) = L^{-1}\{F(s)\} = \frac{1}{5}L^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{5}L^{-1}\left\{\frac{s-1}{(s-1)^2+1}\right\} + \frac{7}{5}L^{-1}\left\{\frac{1}{(s-1)^2+1}\right\} = \frac{1}{5}e^{-t} - \frac{1}{5}e^t\cos t + \frac{7}{5}e^t\sin t$.