Hw2 Solution

Section 2.1 9. Solution:

(1) $p(t) = 1,\ g(t) = 2te^{2t}$

(2) $\mu = e^{\int p(t)dt} = e^{\int 1dt} = e^{-t}$

(3) $\int \mu(t)g(t)dt = \int e^{-t}2te^{2t}dt = 2te^t - 2e^t$

(4) $y(t) = \frac{\int \mu(t)g(t)dt}{\mu(t)} = \frac{2te^t-2e^t+c}{e^{-t}} = 2(t-1)e^{2t} + ce^t.$

(5) $y(0) = 1 \Longrightarrow 2(0-1)e^0 + c = 1 \Longrightarrow c = 3 \Longrightarrow y(t) = 2(t-1)e^{2t} + 3e^t.$

20. Solution:

(1) $p(t) = -1,\ g(t) = 1 + 3\sin t$

(2) $\mu = e^{\int p(t)dt} = e^{\int -1dt} = e^{-t}$

(3) $\int \mu(t)g(t)dt = \int e^{-t}(1+3\sin t)dt = \left(-1 - \frac{3\cos t}{2} - \frac{3\sin t}{2}\right)e^{-t} + c$

(4)

$$\begin{aligned} y(t) &= \frac{\int \mu(t)g(t)dt}{\mu(t)} \\ &= \frac{\left(-1 - \frac{3\cos t}{2} - \frac{3\sin t}{2}\right)e^{-t} + c}{e^{-t}} \\ &= -1 - \frac{3\cos t}{2} - \frac{3\sin t}{2} + ce^t. \end{aligned}$$

When $c = 0$, the solution remains finite as $t \to +\infty$.
$y_0 = y(0) = -1 - \frac{3\cos 0}{2} - \frac{3\sin 0}{2} = -1 - 3/2 = -5/2.$

21. Solution:

(1) $p(t) = -\frac{3}{2},\ g(t) = 3t + 2e^t$

(2) $\mu = e^{\int p(t)dt} = e^{\int -\frac{3}{2}dt} = e^{-3/2t}$

(3) $\int \mu(t)g(t)dt = \int e^{-3/2t}(3t+2e^t)dt = -2te^{-3/2t} - 4/3e^{-3/2t} - 4e^{-t/2} + c$

(4)

$$\begin{aligned} y(t) &= \frac{\int \mu(t)g(t)dt}{\mu(t)} \\ &= \frac{-2te^{-3/2t} - 4/3e^{-3/2t} - 4e^{-t/2} + c}{e^{-3/2t}} \\ &= -2t - 4/3 - 4e^t + ce^{3/2t}. \end{aligned}$$

When $c = 0$, the solution separate solutions that grow positively as $t \to +\infty$ from those that grow negatively.
$y_0 = y(0) = -2 \cdot 0 - 4/3 - 4e^0 + 0 = -4/3 - 4 = -16/3$.
This special solution is $y(t) = -2t - 4/3 - 4e^t$. $y(t) \to -\infty$ as $t \to +\infty$.

Section 2.2 2.

$$y' + y^2 \sin x = 0$$

if $y \neq 0$

(1) $-y^{-2}dy = \sin x dx$

(2) $-\int y^{-2}dy = \int \sin x dx$

(3) $y^{-1} = -\cos x + c$ if $y \neq 0$

$y = 0$ everywhere is a solution as well.

6.

$$\frac{dy}{dx} = \frac{x^2}{1+y^2}$$

(1) $(1+y^2)dy = x^2 dx$

(2) $\int 1 + y^2 dy = \int x^2 dx$

(3) $y + y^3/3 = x^3/3 + c$

18.

$$y' = \frac{3x^2}{3y^2 - 4}$$

(1) $(3y^2 - 4)dy = 3x^2 dx$

(2) $\int 3y^2 - 4dy = \int 3x^2 dx$

(3) $y^3 - 4y = x^3 + c \quad (3y^2 - 4 \neq 0)$

(4) $y(1) = 0 \Longrightarrow 0 = 1 + c \Longrightarrow c = -1 \Longrightarrow$

$$y^3 - 4y = x^3 - 1. \tag{1}$$

If $3y^2 - 4 = 0 \Longrightarrow y = 2/\sqrt{3}$ or $y = -2/\sqrt{3}$.

When $y = 2/\sqrt{3}$, from (1) we have $x^3 = y^3 - 4y + 1 = \left(2/\sqrt{3}\right)^3 - 8/\sqrt{3} + 1 \Longrightarrow$ $x \approx -1.28$.

When $y = -2/\sqrt{3}$, from (1) we have $x^3 = y^3 - 4y + 1 = -\left(2/\sqrt{3}\right)^3 + 8/\sqrt{3} + 1 \Longrightarrow$ $x = 1.60$

Therefore, the interval in which the solution is valid is $(-1.28, 1.60)$.

20.

$$y' = \frac{2 - e^x}{3 + 2y}$$

(1) $(3 + 2y)dy = 2 - e^x dx$

(2) $\int 3 + 2y dy = \int 2 - e^x dx$

(3) $3y + y^2 = 2x - e^x + c \quad (3 + 2y \neq 0)$

(4) $y(0) = 0 \Longrightarrow 0 = 0 - 1 + c \Longrightarrow c = 1 \Longrightarrow$

$$3y + y^2 + e^x - 2x - 1 = 0. \tag{2}$$

$\Longrightarrow y = \frac{-3 \pm \sqrt{9 - 4e^x + 8x + 4}}{2} = \frac{-3 \pm \sqrt{13 - 4e^x + 8x}}{2}$.

Since $y(0) = 0$, we have $y = \frac{-3 + \sqrt{13 - 4e^x + 8x}}{2}$.

$(13 - 4e^x + 8x)' = -4e^x + 8 = 0 \rightarrow x = \ln(2)$. Therefore y attains its maximum value when $x = \ln(2)$.