Hw4 Solution

Section 2.3 1. Solution: Let $Q(t)$ be the amount of the dye in the tank at time t. We have

$$\begin{cases} \frac{dQ}{dt} &= -\frac{Q}{200}\cdot 2 \\ Q(0) &= 200\cdot 1 = 200. \end{cases}$$

$$\frac{dQ}{dt} = -\frac{Q}{100} \Longrightarrow \frac{100}{Q}dQ = -dt \Longrightarrow 100\ln Q = -t + C \Longrightarrow Q = e^{\frac{C-t}{100}}.$$

$$Q(0) = 200 \to 200 = e^{\frac{C}{100}} \to C = 100\ln 200$$

Therefore

$$Q(t) = e^{\frac{100\ln 200 - t}{100}} = 200e^{-\frac{t}{100}}.$$

We want that $Q(t) = 1\%Q(0) = 2 \to 200e^{-\frac{t}{100}} = 2 \to t = 100\ln 100 \approx 460.5$.

7. Solution: Let $B(t)$ be the balance the student owes at time t. We have

$$\begin{cases} \frac{dB}{dt} &= 0.1B - k \\ B(0) &= 8000. \end{cases}$$

To pay off the loan in 3 years, we have $B(3) = 0$.

$$\frac{dB}{dt} - 0.1B = -k \Longrightarrow$$

(1) $p(t) = -0.1$, $g(t) = -k$

(2) $\mu = e^{\int p(t)dt} = e^{\int -0.1dt} = e^{-0.1t}$

(3) $y(t) = \frac{\int \mu(t)g(t)dt}{\mu(t)} = \frac{\int e^{-0.1t}(-k)dt + c}{e^{-0.1t}} = \frac{10ke^{-0.1t}+c}{e^{-0.1t}} = 10k + ce^{0.1t}$.

$B(0) = 8000 \Longrightarrow 8000 = 10k + c \Longrightarrow c = 8000 - 10k$. Therefore

$$Q(t) = 10k + (8000 - 10k)e^{0.1t}.$$

$B(3) = 0 \Longrightarrow 0 = 10k + (8000 - 10k)e^{0.3} \Longrightarrow k = 3086.64$.

The total interest is paid

$$\begin{aligned} \int_0^3 B(t)0.1dt &= \int_0^3 10k + (8000 - 10k)e^{0.1t}dt \\ &= 3k + (8000 - k)10e^{0.1t}|_{t=0}^{t=3} \\ &= 1259.91. \end{aligned}$$