Hw9 Solution

Section 3.4

7. $16y'' + 24y' + 9y = 0$

Solution:

$16r^2 + 24r + 9 = 0 \Longrightarrow (4r+3)^2 = 0 \Longrightarrow r_1 = r_2 = r = -3/4.$

We have

$$y_1(t) = e^{rt} = e^{-3/4t} \quad y_2(t) = te^{rt} = te^{-3/4t}.$$

The general solution is $\boxed{y = C_1y_1 + C_2y_2 = C_1e^{-3/4t} + C_2te^{-3/4t}.}$

9. $9y'' - 12y' + 4y = 0 \quad y(0) = 2, y'(0) = -1.$

Solution:

$9r^2 - 12r + 4 = 0 \Longrightarrow (3r-2)^2 = 0 \Longrightarrow r_1 = r_2 = r = 2/3.$

We have

$$y_1(t) = e^{rt} = e^{2/3t} \quad y_2(t) = te^{rt} = te^{2/3t}.$$

The general solution is $\boxed{y = C_1y_1 + C_2y_2 = C_1e^{2/3t} + C_2te^{2/3t}.}$

Therefore

$$y'(t) = 2/3C_1e^{2/3t} + C_2e^{2/3t} + 2/3C_2te^{2/3t}.$$

$$\begin{aligned} y(0) = 2 &\implies C_1 = 2 \\ y'(0) = -1 &\implies 2/3C_1 + C_2 = -1 \\ &\implies \\ & \quad C_1 = 2 \quad C_2 = -7/3. \end{aligned}$$

$$\boxed{y(t) = 2e^{2/3t} - 7/3te^{2/3t} \quad y \to -\infty \text{ as } t \to +\infty.}$$

Section 3.5

3. $y'' + y' - 6y = 12e^{3t} + 12e^{-2t}$

Solution:

$r^2 + r - 6 = 0 \Longrightarrow r_1 = -3,\ r_2 = 2.$

We have

$$y_1(t) = e^{r_1t} = e^{-3t} \quad y_2(t) = e^{r_2t} = e^{2t}.$$

The general solution is $\boxed{y = C_1y_1 + C_2y_2 = C_1e^{-3t} + C_2e^{2t}.}$

Let $Y_1(t) = Ae^{3t}$ be a particular solution of $y'' + y' - 6y = 12e^{3t}$.

$$Y_1' = 3Ae^{3t} \quad Y_1'' = 9Ae^{3t}$$

$$Y_1'' + Y_1' - 6Y_1 = 9Ae^{3t} + 3Ae^{3t} - 6Ae^{3t} = 6Ae^{3t} = 12e^{3t} \Longrightarrow A = 2.$$

Let $Y_2(t) = Ae^{-2t}$ be a particular solution of $y'' + y' - 6y = 12e^{3t}$.

$$Y_2' = -2Ae^{-2t} \quad Y_2'' = 4Ae^{-2t}$$

$$Y_2'' + Y_2' - 6Y_2 = 4Ae^{-2t} - 2Ae^{-2t} - 6Ae^{-2t} = -4Ae^{3t} = 12e^{-2t} \Longrightarrow A = -3.$$

Therefore the general solution is $\boxed{y(t) = C_1e^{-3t} + C_2e^{2t} + 2e^{3t} - 3e^{-2t}}$

4. $y'' - 2y' - 3y = -3te^{-t}$
   Solution:
   $r^2 - 2r - 3 = 0 \Longrightarrow r_1 = 3,\ r_2 = -1.$
   We have

$$y_1(t) = e^{r_1t} = e^{3t} \quad y_2(t) = e^{r_2t} = e^{-t}.$$

The general solution is $\boxed{y = C_1y_1 + C_2y_2 = C_1e^{3t} + C_2e^{-t}.}$
Let $Y(t) = t(At + B)e^{-t}$ be a particular solution of $y'' - 2y' - 3y = -3te^{-t}$.

$$\begin{aligned} Y' &= 2Ae^{-t} - At^2e^{-t} + Be^{t} - tBe^{-t} \\ Y'' &= 2Ae^{-t} - 2Ate^{-t} - 2Ate^{-t} + At^2e^{-t} - Be^{-t} - Be^{-t} + tBe^{-t} \\ &= 2Ae^{t} - 4Ate^{-t} + At^2e^{-t} - 2Be^{-t} + tBe^{-t}. \end{aligned}$$

Therefore

$$\begin{aligned} Y'' - 2Y' - 3Y &= 2Ae^{-t} - 4Ate^{-t} + At^2e^{-t} - 4Ate^{-t} + 2Att^2 - 3At^2e^{-t} \\ &\quad -2Be^{-t} + tBe^{-t} - 2Be^{-t} + 2tBe^{-t} - 3tBe^{-t} \\ &= (2A - 4B)e^{-t} - 8Ate^{-t} = -3te^{-t}. \\ &\Longrightarrow \\ &\quad 2A - 4B = 0 \quad -8A = -3 \\ &\Longrightarrow \\ &\quad A = 3/8 \quad B = 3/16. \end{aligned}$$

Therefore the general solution is $\boxed{y(t) = C_1e^{3t} + C_2e^{-t} + 3/8t^2e^{-t} + 3/16te^{-t}}$

11. $y'' + y' - 2y = 2t \quad y(0) = 0, y'(0) = 1$

Solution:

$r^2 + r - 2 = 0 \Longrightarrow r_1 = -2,\ r_2 = 1.$

We have

$$y_1(t) = e^{r_1 t} = e^{-2t} \quad y_2(t) = e^{r_2 t} = e^t.$$

The general solution is $\boxed{y = C_1 y_1 + C_2 y_2 = C_1 e^{-2t} + C_2 e^t.}$

Let $Y(t) = (At + B)$ be a particular solution of $y'' + y' - 2y = 2t$.

$$\begin{aligned} Y' &= A \\ Y'' &= 0. \end{aligned}$$

Therefore

$$\begin{aligned} Y'' + Y' - 2Y &= 0 + A - 2(At + B) = -2At + A - 2B = 2t \\ &\Longrightarrow \\ &-2A = 2 \quad A - 2B = 0 \\ &\Longrightarrow \\ &A = -1 \quad B = -1/2. \end{aligned}$$

Therefore the general solution is $\boxed{y(t) = C_1 e^{-2t} + C_2 e^t - t - 1/2}$

$$y'(t) = -2C_1 e^{-2t} + C_2 e^t - 1.$$

$$\begin{aligned} y(0) = 0 &\Longrightarrow C_1 + C_2 - 1/2 = 0 \\ y'(0) = 1 &\Longrightarrow -2C_1 + C_2 - 1 = 1 \\ &\Longrightarrow \\ &C_1 = -1/2 \quad C_2 = 1. \end{aligned}$$

The solution is $\boxed{y(t) = -1/2 e^{-2t} + e^t - t - 1/2}$