

Problem 1 (20 points) In each of the following statements, determine true or false.

- (a) The minor M_{11} of $\begin{bmatrix} -3 & 4 & 2 \\ 6 & 3 & 0 \\ 4 & 0 & 1 \end{bmatrix}$ is 3 and the cofactor C_{11} is -3 .
- (b) The matrix B is obtained by multiplying a row of the matrix A with 2 and adding to another row. Then the determinants of A and B are equal.
- (c) If a matrix A is singular, then the solution of $\underline{x} = 0$ is the only solution of $Ax = 0$.
- (d) S is a set of a vector space V . S is a basis of V if V is spanned by S .
- (e) The set of all 2×2 matrices of the form $\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$ with the standard operations is a vector space.

True: b e

False: a c d

Problem 2 (10 points) Use a determinant to find the area of the triangle with the given vertices $(1, 0)$, $(5, 0)$, and $(5, 8)$.

Solution. $\text{Area} = \pm \frac{1}{2} \left| \begin{array}{ccc} 1 & 0 & 1 \\ 5 & 0 & 1 \\ 5 & 8 & 1 \end{array} \right| = \frac{1}{2} \left(40 - 8 \right) = \frac{1}{2} 32 = 16$

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Problem 3 (20 points) Let $A = \begin{bmatrix} 1 & 4 & -2 \\ 3 & 2 & 0 \\ -1 & 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Calculate $\det A$, $\det B$, $\det AB$, and $\det B'A'$.

$$\text{Solution } \det A = \begin{vmatrix} 1 & 4 & -2 \\ 3 & 2 & 0 \\ -1 & 4 & 3 \end{vmatrix} = 6 + 0 - 24 - 4 - 36 = -58 \quad 7$$

$$\det B = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 2 \quad 7$$

$$AB = \begin{bmatrix} 1 & 4 & -2 \\ 3 & 2 & 0 \\ -1 & 4 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 3 \\ 6 & 5 & 5 \\ -2 & 3 & 6 \end{bmatrix}$$

$$\det AB = \begin{vmatrix} 2 & 5 & 3 \\ 6 & 5 & 5 \\ -2 & 3 & 6 \end{vmatrix} = 60 - 50 + 54 + 30 - 30 - 180 = -116$$

$$\text{or } \det(AB) = \det A \cdot \det B = -58 \cdot 2 = -116 \quad 3$$

$$\det(B'A') = \det(AB)^T = \det AB = -116 \quad 3$$

Problem 4 (15 points)

Solving the system of linear equations by the Cramer's Rule (no credit if you use any other methods).

$$\begin{cases} 3x + 3y + 5z = 1, \\ 3x + 5y + 9z = 2, \\ 5x + 9y + 17z = 4. \end{cases}$$

Solution.

$$A = \begin{bmatrix} 3 & 3 & 5 \\ 3 & 5 & 9 \\ 5 & 9 & 17 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \quad \det A = \begin{vmatrix} 3 & 3 & 5 \\ 3 & 5 & 9 \\ 5 & 9 & 17 \end{vmatrix} = 4$$

$$\det A_1 = \begin{vmatrix} 1 & 3 & 5 \\ 2 & 5 & 9 \\ 4 & 9 & 17 \end{vmatrix} = 0 \quad \det A_2 = \begin{vmatrix} 3 & 1 & 5 \\ 3 & 2 & 9 \\ 5 & 4 & 17 \end{vmatrix} = -2$$

$$\det A_3 = \begin{vmatrix} 3 & 3 & 1 \\ 3 & 5 & 2 \\ 5 & 9 & 4 \end{vmatrix} = 2$$

$$x = \frac{\det A_1}{\det A} = 0 \quad y = \frac{\det A_2}{\det A} = \frac{-2}{4} = -\frac{1}{2} \quad z = \frac{\det A_3}{\det A} = \frac{2}{4} = \frac{1}{2}$$

Problem 5 (20 points)

Is S a basis for \mathbb{R}^3 ?

$$S = \{(4, 0, 1), (0, -3, 2), (5, 10, 0)\}.$$

Solution: 1. given any vector (x, y, z) in \mathbb{R}^3

$$(x, y, z) = C_1(4, 0, 1) + C_2(0, -3, 2) + C_3(5, 10, 0)$$

$$= (4C_1 + 5C_3, -3C_2 + 10C_3, C_1 + 2C_2)$$

$$\Rightarrow \begin{cases} 4C_1 + 5C_3 = x \\ -3C_2 + 10C_3 = y \\ C_1 + 2C_2 = z \end{cases}$$

Coefficient matrix $A = \begin{bmatrix} 4 & 0 & 5 \\ 0 & -3 & 10 \\ 1 & 2 & 0 \end{bmatrix}$.

$$\det A = -15 - 80 = -95 \neq 0$$

\Rightarrow has a unique solution. S spans \mathbb{R}^3

2. $C_1(4, 0, 1) + C_2(0, -3, 2) + C_3(5, 10, 0) = 0$ 2

$$\begin{cases} 4C_1 + 5C_3 = 0 \\ -3C_2 + 10C_3 = 0 \\ C_1 + 2C_2 = 0 \end{cases}$$

Coefficient matrix A is nonsingular

$$C_1 = C_2 = C_3 = 0$$

S is linearly independent

therefore S is a basis for \mathbb{R}^3

Problem 6 (15 points) Which of the subsets of R^3 is a subspace of R^3 ? Show your work.

$$1. W = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 0\}$$

$$2. W = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 1\}$$

Solution: 1. $U = (x_1, x_2, x_3), V = (y_1, y_2, y_3) \in W$ $x_1 + x_2 + x_3 = 0, y_1 + y_2 + y_3 = 0$

$$u+v = (x_1+y_1, x_2+y_2, x_3+y_3)$$

$$x_1+y_1+x_2+y_2+x_3+y_3 = (x_1+x_2+x_3) + (y_1+y_2+y_3)$$

$$= 0+0=0$$

Therefore $u+v \in W$

$$cU = (cx_1, cx_2, cx_3) \quad cx_1 + cx_2 + cx_3 = c(x_1 + x_2 + x_3)$$

$$cu \in W$$

$$= c \cdot 0 = 0$$

W is a subspace

$$2. U = (1, 0, 0) \quad V = (0, 1, 0) \quad u, v \in W$$

$$u+v = (1, 1, 0) \notin W$$

W is not a subspace.

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