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**Problem 1 (20 points)** In each of the following statements, determine true or false.

- (a)  $e^2x + \sin(2)y = 0$  is a linear equation in the variables  $x$  and  $y$ .
- (b)  $2xy + 3y - 1 = 0$  is a linear equation in the variables  $x$  and  $y$ .
- (c) The system of linear equations  $\begin{cases} 3x + y = 0, \\ 6x + 2y = 1. \end{cases}$  has no solution.
- (d) The system of linear equations  $\begin{cases} 3x + y = 0, \\ 2x + 2y = 1. \end{cases}$  has no solution.
- (e) The size of the matrix  $A$  is  $n \times 2$  and  $B$  is  $n \times 1$ . The size of the matrix  $(A^T A)^{-1} A^T B$  is  $2 \times n$ .

True: (a) (c)

False: (b) (d) (e)

**Problem 2 (20 points)**

Let  $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ . Calculate  $4A$ ,  $AA^T$ ,  $A^T A$ , and  $(AA^T)^{-1}$ .

Solution  $4A = 4 \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 4 \\ 0 & 8 & 4 \end{bmatrix} \quad (1)$

$$AA^T = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 5 & 5 \end{bmatrix} \quad (2)$$

$$A^T A = \begin{bmatrix} 1 & 0 \\ 2 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 8 & 4 \\ 1 & 4 & 2 \end{bmatrix} \quad (3)$$

$$(AA^T)^{-1} = \begin{bmatrix} 6 & 5 \\ 5 & 5 \end{bmatrix}^{-1} = \frac{1}{30-25} \begin{bmatrix} 5 & -5 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1.2 \end{bmatrix} \quad (4)$$

**Problem 3 (15 points)** Write the column matrix  $b$  as a linear combination of the columns of  $A$ , where  $A = \begin{bmatrix} -3 & 5 \\ 3 & 4 \\ 4 & -8 \end{bmatrix}$  and  $b = \begin{bmatrix} -22 \\ 4 \\ 32 \end{bmatrix}$ .

Solution

$$A\mathbf{x} = b \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\left[ \begin{array}{cc|c} -3 & 5 & -22 \\ 3 & 4 & 4 \\ 4 & -8 & 32 \end{array} \right] \xrightarrow{\text{(1)}} \left[ \begin{array}{cc|c} -3 & 5 & -22 \\ 0 & 9 & -18 \\ 0 & -\frac{4}{3} & \frac{8}{3} \end{array} \right] \xrightarrow{\text{(2)}} \left[ \begin{array}{cc|c} -3 & 5 & -22 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{array} \right]$$

(1) (2) (3)

$$-3x_1 + 5x_2 = -22 \quad -3x_1 = -22 - 5x_2 \Rightarrow -22 - 5(-2) = -12$$

$$x_2 = -2 \quad (2) \qquad \Rightarrow x_1 = 4 \quad (3)$$

$$\begin{bmatrix} -22 \\ 4 \\ 32 \end{bmatrix} = -4 \begin{bmatrix} -3 \\ 3 \\ 4 \end{bmatrix} + (-2) \begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix}$$

(1) (2) (3)

**Problem 4 (20 points)**

Consider the system of linear equations

$$\begin{cases} 2x + y + 2z = 4, \\ 2x + 2y = 5, \\ 2x - y + 6z = 2. \end{cases}$$

- (a) Write down the augmented matrix;  
 (b) Find the solutions using Gaussian Elimination and back-substitution.

Solution:

(a) The augmented matrix

$$\left[ \begin{array}{cccc} 2 & 1 & 2 & 4 \\ 2 & 2 & 0 & 5 \\ 2 & -1 & 6 & 2 \end{array} \right] \textcircled{5}$$

$$(b) \left[ \begin{array}{cccc} 2 & 1 & 2 & 4 \\ 2 & 2 & 0 & 5 \\ 2 & -1 & 6 & 2 \end{array} \right] \rightarrow \left[ \begin{array}{cccc} 2 & 1 & 2 & 4 \\ 0 & 1 & -2 & 1 \\ 0 & -2 & 4 & -2 \end{array} \right] \rightarrow \left[ \begin{array}{cccc} 2 & 1 & 2 & 4 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \textcircled{8}$$

$$\begin{aligned} 2x + y + 2z &= 4 & z = t & \textcircled{1} \\ y - 2z &= 1 & y = 1 + 2t & \textcircled{2} \\ 2x - y - 2z &= 4 - 1 - 2t - 2t = 3 - 4t & & \\ x &= \frac{3}{2} - 2t & & \textcircled{3} \end{aligned}$$

$$\begin{cases} x = \frac{3}{2} - 2t \\ y = 1 + 2t \\ z = t \end{cases} \textcircled{2}$$

### Problem 5 (20 points)

A hardware retailer wants to know the demand for a rechargeable power drill as a function of price. The table shows the monthly sales  $y$  for three different prices

Price $x$	\$25	\$30	\$35
Demand $y$	82	75	67

(a) Fit the data with a quadratic polynomial.

(b) Find the least square regression line for the data.

Solution. (a)  $z = x - 30$

$$p(z) = a_0 + a_1 z + a_2 z^2 \quad (2)$$

$$\text{or } p(x) = a_0 + a_1 x + a_2 x^2$$

$$\Rightarrow a_0 = 102, a_1 = -0.3, a_2 = -0.02$$

$$p(x) = 102 - 0.3x - 0.02x^2$$

$$p(x) = 82 - 1.3(x-25) - 0.02(x-25)^2$$

$$p(-5) = 82 \Rightarrow a_0 + (-5)a_1 + 25a_2 = 82 \quad (1)$$

$$p(0) = 75 \Rightarrow a_0 = 75 \quad (2)$$

$$p(5) = 67 \Rightarrow a_0 + 5a_1 + 25a_2 = 67 \quad (3)$$

$$\begin{cases} a_0 = 75 \\ a_1 = -\frac{1}{50} \\ a_2 = -\frac{3}{2} \end{cases}$$

$$p(x) = 75 - \frac{1}{2}(x-30) - \frac{1}{50}(x-30)^2 \quad (2)$$

$$(b) \quad Y = \begin{bmatrix} 82 \\ 75 \\ 67 \end{bmatrix} \quad X = \begin{bmatrix} 1 & 25 \\ 1 & 30 \\ 1 & 35 \end{bmatrix} \quad X^T X = \begin{bmatrix} 1 & 1 & 1 \\ 25 & 30 & 35 \end{bmatrix} \begin{bmatrix} 1 & 25 \\ 1 & 30 \\ 1 & 35 \end{bmatrix} = \begin{bmatrix} 3 & 90 \\ 90 & 2750 \end{bmatrix} \quad (2)$$

$$(X^T X)^{-1} X^T Y = \frac{1}{8250-8100} \begin{bmatrix} 2800 & -90 \\ -90 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 25 & 30 & 35 \end{bmatrix} \begin{bmatrix} 82 \\ 75 \\ 67 \end{bmatrix}$$

$$= \frac{1}{150} \begin{bmatrix} 2800 & -90 \\ -90 & 3 \end{bmatrix} \begin{bmatrix} 224 \\ 6645 \end{bmatrix}$$

$$= \frac{1}{150} \begin{bmatrix} 718 \\ -9 \end{bmatrix} = \begin{bmatrix} 1.1967 \\ -0.06 \end{bmatrix} \quad (2) = \begin{bmatrix} \frac{359}{3} \\ -0.06 \end{bmatrix}$$

The least square regression line  $\Rightarrow 1.1967 - 0.06x$