Name:____________________ **Total:**____________________

Math 320 Midterm Exam I (09/25/2019)

1. (15 points) In each of the following problems determine the order of the given differential equation, also state whether the equation is linear or nonlinear. 1.5 each

   (a) $3y'y = 2t^2$; first order nonlinear

   (b) $(\sqrt{3}-1)t^3y'' + t^2y' - 3y + \cos t = 0$; Second order linear

   (c) $ty' + e^t = 0$; first order linear

   (d) $y'' + \cos y = 0$; Second order nonlinear

   (e) $2x''' - 7xx' + x' + 9x = 0$. third order nonlinear

2. (15 points) Find the solution of the initial value problem

$$y' = -4y + te^{-2t}, \quad y(0) = 1.$$

Determine the behavior of the solution as $t \to +\infty$.

Solution $y' + 4y = te^{-2t}$

① $p(t) = 4$, ① $g(t) = te^{-2t}$ ①

② $\mu(t) = e^{\int p(t)dt} = e^{4t}$ ②

③ $\int \mu(t)g(t)dt = \int e^{4t}\cdot te^{-2t}dt = \int te^{2t}dt = \frac{t}{2}e^{2t} - \frac{1}{2}\int e^{2t}dt = \frac{t}{2}e^{2t} - \frac{1}{4}e^{2t}$ ③

④ $y(t) = \dfrac{\int \mu(t)g(t)dt + C}{\mu(t)} = \dfrac{\frac{t}{2}e^{2t} - \frac{1}{4}e^{2t} + C}{e^{4t}}$ ① $= \frac{t}{2}e^{-2t} - \frac{1}{4}e^{-2t} + Ce^{-4t}$ ②

$y(0) = 1 \Rightarrow 0 - \frac{1}{4} + C = 1 \Rightarrow C = \frac{5}{4}$ ②

$y(t) = \frac{t}{2}e^{-2t} - \frac{1}{4}e^{-2t} + \frac{5}{4}e^{-4t}$ ①

$t \to +\infty \quad y(t) \to 0$ ②

3. (15 points) Find the general solution of the differential equation

$$y' + y^2 \sin(t) = 0.$$

And find the special solution satisfying the initial condition $y(\frac{\pi}{2}) = 1$.

Solution:

$$\frac{dy}{dt} = -y^2 \operatorname{Sin}(t)$$

5 $$\frac{dy}{y^2} = -\operatorname{Sin} t \, dt$$

$$\int \frac{dy}{y^2} = \int -\operatorname{Sin} t \, dt$$

3 $$-\frac{1}{y} = \cos t + c$$

2 $$y = \frac{-1}{\cos t + c}$$

$$y(\tfrac{\pi}{2}) = 1 \implies 1 = \frac{-1}{\cos\frac{\pi}{2} + c} = \frac{-1}{c} \implies c = -1$$ 3

$$y(t) = \frac{-1}{\cos t - 1}$$ 2

4. (20 points) Determine whether the following equation is exact or not. If it is exact, find the solution.

$$(te^t + y - 1)dt + (2y + t)dy = 0.$$

Solution

$$M(t,y) = te^t + y - 1 \quad 2$$

$$N(t,y) = 2y + t \quad 2$$

$$\frac{\partial M}{\partial y} = 1 \quad 2 \qquad \frac{\partial N}{\partial t} = 1 = \frac{\partial M}{\partial y} \quad 2$$

the equation is exact 1

$$\psi = \int M dt = \int te^t + y - 1\, dt = te^t - e^t + (y-1)t + C_1(y) \quad 2$$

$$\psi = \int N dy = \int 2y + t\, dy = y^2 + ty + C_2(t) \quad 2$$

$$\Rightarrow C_1(y) = y^2. \qquad C_2(t) = te^t - e^t - t \quad 1$$

$$\psi = te^t - e^t + (y-1)t + y^2 \quad 2$$

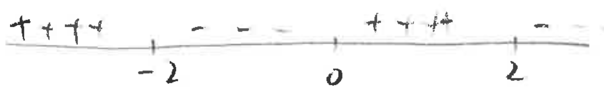
the solution is $\quad te^t - e^t + (y-1)t + y^2 = C$

5. (15 points) Determine the steady states and classify each one as asymptotically stable or unstable for the differential equation $\frac{dy}{dt} = y(4 - y^2)$. Draw the phase line and sketch several graphs of solutions.

Solution. $f(y) = y(4-y^2)$

$f(y)=0 \Rightarrow y=0,\ y=-2,\ y=2$

the steady states are $0,\ -2,\ 2$ 3



4

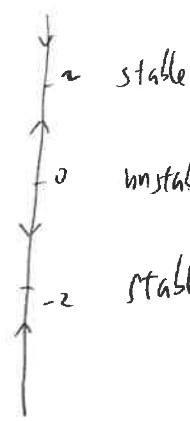
$y<-2,\quad f(y)>0$

$-2<y<0,\quad f(y)<0$

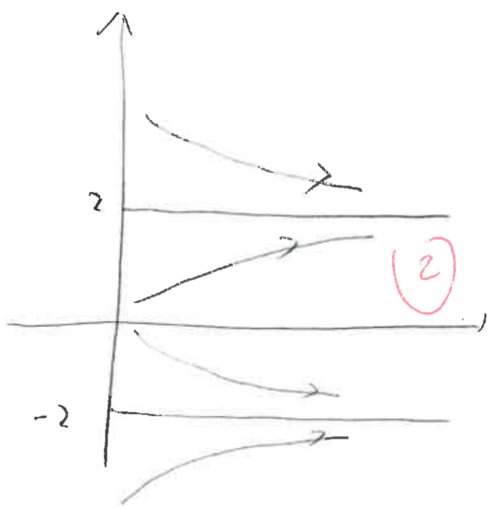
$0<y<2\quad f(y)>0$

$y>2\quad f(y)<0$

the phase line



3 3



2

6. (20 points) A person wants to borrow $200,000 to buy a house. The lender will charge interest at an annual rate of 9%. Assuming that interest is compounded continuously and that the borrower makes payments continuously at a constant monthly rate k. Determine the payment rate k that is required to pay off the loan in 30 years.

Solution: Let $B(t)$ be the amount the person own the bank at time t

$$\begin{cases} \frac{dB}{dt} = B(t)\frac{0.09}{12} - k \quad 5 \\ B(0) = 200000 \quad 2 \end{cases}$$

$$\frac{dB}{dt} - \frac{0.03}{4}B = -k$$

$$p(t) = -\frac{0.03}{4}, \quad g(t) = -k$$

$$\mu(t) = e^{-\frac{0.03}{4}t} \qquad \int g(t)\mu(t)\,dt = -\frac{4}{0.03}e^{-\frac{0.03}{4}t}k$$

$$B(t) = \frac{-\frac{4}{0.03}e^{-\frac{0.03}{4}t}k + c}{e^{-\frac{0.03}{4}t}} = \frac{4k}{0.03} + ce^{\frac{0.03}{4}t}$$

$$B(0) = 200000 \Rightarrow \frac{4k}{0.03} + c = 200000 \Rightarrow c = 200000 - \frac{4k}{0.03}$$

$$B(t) = \frac{4k}{0.03} + \left(200000 - \frac{4k}{0.03}\right)e^{\frac{0.03}{4}t} \quad 5$$

$$2 \quad B(360) = 0 \Rightarrow \frac{4k}{0.03} + \left(200000 - \frac{4k}{0.03}\right)e^{\frac{0.03}{4}\cdot 360} = 0 \quad 5$$

$$\frac{4k}{0.03}\left(e^{\frac{0.03}{4}\cdot 360} - 1\right) = 200000\cdot e^{\frac{0.03}{4}\cdot 600}$$

$$k = 1608.07$$

1

annual rate $1.929684\cdot 10^4$

56
1W

858 V8

290	16
280	3
270	2
260	2
250	1