Hw1 Solution

Section 1.3

1. Second order, linear.
3. Fourth order, linear.
4. Second order, nonlinear.
7. Solution:
   $y = 3t + t^2$, $y' = 3 + 2t$
   $ty' - y = t(3 + 2t) - (3t + t^2) = 3t + 2t^2 - 3t - t^2 = t^2$.
9. Solution:
   $y_1 = t^{-2}$, $y_1' = -2t^{-3}$, $y_1'' = 6t^{-4}$
   $t^2 y_1'' + 5ty_1' + 4y_1 = t^2(6t^{-4}) + 5t(-2t^{-3}) + 4t^{-2} = 6t^{-2} - 10t^{-2} + 4t^{-2} = 0$.

   $y_2 = t^{-2}\ln t$, $y_2' = -2t^{-3}\ln t + t^{-2}/t = -2t^{-3}\ln t + t^{-3}$,
   $y_2'' = 6t^{-4}\ln t - 2t^{-4} - 3t^{-4} = 6t^{-4}\ln t - 5t^{-4}$
   $t^2y_2'' + 5ty_2' + 4y_2 = t^2(6t^{-4}\ln t - 5t^{-4}) + 5t(-2t^{-3}\ln t + t^{-3}) + 4t^{-2}\ln t = 6t^{-2}\ln t - 10t^{-2}\ln t + 4t^{-2}\ln t - 5t^{-2} + 5t^{-2} = 0$.
12. Solution:
   $y = e^{rt}$, $y' = re^{rt}$, $y'' = r^2e^{rt}$
   $y'' + y' - 6y = r^2e^{rt} + re^{rt} - 6e^{rt} = (r^2 + r - 6)e^{rt} = 0 \Longrightarrow r^2 + r - 6 = 0 \Longrightarrow r = 2$ or $r = -3$.
15. Solution:
   $y = t^r$, $y' = rt^{r-1}$, $y'' = r(r-1)t^{r-2}$
   $t^2y'' - 4ty' + 4y = t^2r(r-1)t^{r-2} - 4trt^{r-1} + 4t^r = (r(r-1) - 4r + 4)t^r = 0 \Longrightarrow r^2 - 5r + 4 = 0 \Longrightarrow r = 1$ or $r = 4$.

Section 2.1

5c. Solution:
   (1) $p(t) = -2$, $g(t) = 3e^t$
   (2) $\mu = e^{\int p(t)dt} = e^{\int -2dt} = e^{-2t}$
   (3) $\int \mu(t)g(t)dt = \int e^{-2t}3e^t dt = -3e^{-t}$
   (4) $y(t) = \frac{\int \mu(t)g(t)dt}{\mu(t)} = \frac{-3e^{-t}+c}{e^{-2t}} = -3e^t + ce^{2t}$.

When $t \to \infty$, $y(t) \to +\infty$ if $c > 0$ or $y(t) \to -\infty$ if $c \le 0$

9. Solution:

(1) $p(t) = 1,\ g(t) = 2te^{2t}$

(2) $\mu = e^{\int p(t)dt} = e^{\int 1 dt} = e^{-t}$

(3) $\int \mu(t)g(t)dt = \int e^{-t}2te^{2t}dt = 2te^{t} - 2e^{t}$

(4) $y(t) = \frac{\int \mu(t)g(t)dt}{\mu(t)} = \frac{2te^{t}-2e^{t}+c}{e^{-t}} = 2(t-1)e^{2t} + ce^{t}$.

(5) $y(0) = 1 \Longrightarrow 2(0-1)e^{0} + c = 1 \Longrightarrow c = 3 \Longrightarrow y(t) = 2(t-1)e^{2t} + 3e^{t}$.

11. Solution:

(1) $p(t) = \frac{2}{t},\ g(t) = \frac{\cos t}{t^2}$

(2) $\mu = e^{\int p(t)dt} = e^{\int \frac{2}{t}dt} = e^{2\ln t} = t^2$

(3) $\int \mu(t)g(t)dt = \int t^2(\frac{\cos t}{t^2}))dt = \sin t$

(4) $y(t) = \frac{\int \mu(t)g(t)dt}{\mu(t)} = \frac{\sin t + c}{t^2} = \frac{\sin t}{t^2} + ct^{-2}$.

(5) $y(\pi) = 0 \Longrightarrow 0 + c\pi^{-2} = 0 \Longrightarrow c = 0 \Longrightarrow y(t) = \frac{\sin t}{t^2}$ for $t > 0$.

21. Solution:

(1) $p(t) = -\frac{3}{2},\ g(t) = 3t + 2e^{t}$

(2) $\mu = e^{\int p(t)dt} = e^{\int -\frac{3}{2}dt} = e^{-3/2t}$

(3) $\int \mu(t)g(t)dt = \int e^{-3/2t}(3t + 2e^{t})dt = -2te^{-3/2t} - 4/3e^{-3/2t} - 4e^{-t/2}$

(4)

$$\begin{aligned} y(t) &= \frac{\int \mu(t)g(t)dt}{\mu(t)} \\ &= \frac{-2te^{-3/2t} - 4/3e^{-3/2t} - 4e^{-t/2} + c}{e^{-3/2t}} \\ &= -2t - 4/3 - 4e^{t} + ce^{3/2t}. \end{aligned}$$

When $c = 0$, the solution separate solutions that grow positively as $t \to +\infty$ from those that grow negatively.
$y_0 = y(0) = -2 \cdot 0 - 4/3 - 4e^{0} + 0 = -4/3 - 4 = -16/3$.
This special solution is $y(t) = -2t - 4/3 - 4e^{t}$. $y(t) \to -\infty$ as $t \to +\infty$.