Hw10 Solution

Section 7.5 3. $A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$

Solution:

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = \lambda^2 - 4 + 3 = \lambda^2 - 1 = 0 \rightarrow \lambda_1 = 1, \quad \lambda_2 = -1,$$

For $\lambda_1 = 1$

$$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Longrightarrow x_1 - x_2 = 0.$$

The eigenvector is $\xi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For $\lambda_1 = -1$

$$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Longrightarrow 3x_1 - x_2 = 0.$$

The eigenvector is $\xi_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

The general solution is $\boxed{X = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + C_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}}$

6. $A = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}$

Solution:

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 6 \\ -1 & -2-\lambda \end{vmatrix} = (\lambda-3)(\lambda+2)+6 = \lambda^2 - \lambda = 0 \rightarrow \lambda_1 = 1, \; \lambda_2 = 0,$$

For $\lambda_1 = 1$

$$\begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Longrightarrow x_1 + 3x_2 = 0.$$

The eigenvector is $\xi_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

For $\lambda_1 = 0$

$$\begin{bmatrix} 3 & 6 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Longrightarrow x_1 + 2x_2 = 0.$$

The eigenvector is $\xi_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

The general solution is $\boxed{X = C_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^t + C_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$

10. $A = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}, \quad x(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Solution:

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{vmatrix} = (\lambda-5)(\lambda-1)+3 = \lambda^2-6\lambda+8 = 0 \rightarrow \lambda_1 = 2,\ \lambda_2 = 4,$$

For $\lambda_1 = 2$

$$\begin{bmatrix} 3 & -1 \\ 3 & -31 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Longrightarrow 3x_1 - x_2 = 0.$$

The eigenvector is $\xi_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

For $\lambda_1 = 4$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Longrightarrow x_1 - x_2 = 0.$$

The eigenvector is $\xi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The general solution is $\boxed{X = C_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}}$

$$x(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \implies C_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$
$$\implies C_1 + C_2 = 2 \quad 3C_1 + C_2 = -1$$
$$\implies$$
$$C_1 = -3/2 \quad C_2 = 7/2.$$

The solution is $\boxed{X = -3/2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{2t} + 7/2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}. \text{ When } t \to +\infty,\ x(t) \to +\infty.}$

11. $A = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix}, \quad x(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Solution:

$$|A - \lambda I| = \begin{vmatrix} -2-\lambda & 1 \\ -5 & 4-\lambda \end{vmatrix} = (\lambda+2)(\lambda-4)+5 = \lambda^2 - 2\lambda - 3 = 0 \to \lambda_1 = 3,\ \lambda_2 = -1,$$

For $\lambda_1 = 3$

$$\begin{bmatrix} -5 & 1 \\ -5 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \implies -5x_1 + x_2 = 0.$$

The eigenvector is $\xi_1 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

For $\lambda_1 = -1$

$$\begin{bmatrix} -1 & 1 \\ -5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \implies x_1 - x_2 = 0.$$

The eigenvector is $\xi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The general solution is $\boxed{X = C_1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} e^{3t} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}}$

$$x(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \implies C_1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\implies C_1 + C_2 = 1 \quad 5C_1 + C_2 = 3$$

$$\implies$$

$$C_1 = 1/2 \quad C_2 = 1/2.$$

The solution is $\boxed{X = 1/2 \begin{bmatrix} 1 \\ 5 \end{bmatrix} e^{3t} + 1/2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}. \text{ When } t \to +\infty,\ x(t) \to +\infty.}$

Section 7.6 7. $A = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix}, \quad x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Solution:

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -5 \\ 1 & -3-\lambda \end{vmatrix} = (\lambda+1)(\lambda+3)+5 = \lambda^2+2\lambda+2 = 0 \to \lambda = \frac{-2 \pm 2i}{2} = -1 \pm i.$$

For $\lambda_1 = -1 + i$

$$\begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \implies x_1 - (2+i)x_2 = 0.$$

The eigenvector is $\xi_1 = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$.

The general solution is

$$\boxed{X = C_1 e^{-t} \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} \cos t - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin t \right) + C_2 e^{-t} \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} \sin t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos t \right)}$$

$$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \implies C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\implies 2C_1 + C_2 = 1 \quad C_1 = 1$$

$$\implies$$

$$C_1 = 1 \quad C_2 = -1.$$

The solution is

$$X = e^{-t}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\cos t - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin t\right) - e^{-t}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\sin t + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos t\right) = e^{-t}\begin{bmatrix} \cos t - 3\sin t \\ \cos t - \sin t \end{bmatrix}$$

When $t \to +\infty$, $x(t) \to 0$.

8. $A = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix}, \quad x(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Solution:

$$|A - \lambda I| = \begin{vmatrix} -3-\lambda & 2 \\ -1 & -1-\lambda \end{vmatrix} = (\lambda+3)(\lambda+1)+2 = \lambda^2+4\lambda+5 = 0 \to \lambda = \frac{-4 \pm 2i}{2} = -2 \pm i.$$

For $\lambda_1 = -2 + i$

$$\begin{bmatrix} -1-i & 2 \\ -1 & 1-i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Longrightarrow x_1 - (1-i)x_2 = 0.$$

The eigenvector is $\xi_1 = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$.

The general solution is

$$X = C_1 e^{-2t}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos t - \begin{bmatrix} -1 \\ 0 \end{bmatrix}\sin t\right) + C_2 e^{-2t}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\sin t + \begin{bmatrix} -1 \\ 0 \end{bmatrix}\cos t\right)$$

$$\begin{aligned} x(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} &\Longrightarrow C_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ &\Longrightarrow C_1 - C_2 = 1 \quad C_1 = -2 \\ &\Longrightarrow \\ &\quad C_1 = -2 \quad C_2 = -3. \end{aligned}$$

The solution is

$$X = -2e^{-2t}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos t - \begin{bmatrix} -1 \\ 0 \end{bmatrix}\sin t\right) - 3e^{-2t}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\sin t + \begin{bmatrix} -1 \\ 0 \end{bmatrix}\cos t\right)$$

$$X = e^{-2t}\begin{bmatrix} \cos t - 5\sin t \\ -2\cos t - 3\sin t \end{bmatrix}$$

When $t \to +\infty$, $x(t) \to 0$.