Hw13 Solution

Section 6.2 3. $F(s) = \frac{2}{s^2+3s+4}$

Solution:

$$\begin{aligned} F(s) &= \frac{2}{s^2+3s+4} = \frac{2}{(s+4)(s-1)} \\ &= \frac{A}{s+4} + \frac{B}{s-1} = \frac{A(s-1)+B(s+4)}{(s+4)(s-1)} \\ &= \frac{(A+B)s - A + 4B}{(s+4)(s-1)} \end{aligned}$$

We have $\begin{cases} A+B &= 0 \\ -A+4B &= 2 \end{cases} \implies \begin{cases} A &= 2/5 \\ B &= 2/5 \end{cases}$

Therefore $F(s) = \frac{-2/5}{s+4} + \frac{2/5}{s-1} = \frac{2}{5}\frac{1}{s-1} - \frac{2}{5}\frac{1}{s+4}$.

and

$f(t) = L^{-1}\{F(s)\} = \frac{2}{5}L^{-1}\left\{\frac{1}{s-1}\right\} - \frac{2}{5}L^{-1}\left\{\frac{1}{s+4}\right\} = \frac{2}{5}e^{t} - \frac{2}{5}e^{-4t}$.

14. $y^{(4)} - y = 0 \quad y(0) = 1, y'(0) = 0, y''(0) = 1, y'''(0) = 0.$

Solution:

$L\left(y^{(4)} - y\right) = L(0) = 0.$

Let $F(s) = L(y(t))$, we have $L(y^{(4)}) = s^4L(y) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = s^4F(s) - s^3 - s$

$$\begin{aligned} s^4F(s) - s^3 - s - F(s) = 0 &\implies (s^4-1)F(s) = s^3 + s \\ &\implies F(s) = \frac{s(s^2+1)}{((s^2-1)(s^2+1)} = \frac{s}{(s^2-1)}. \end{aligned}$$

$$\begin{aligned} F(s) &= \frac{s}{(s+1)(s-1)} \\ &= \frac{A}{s+1} + \frac{B}{s-1} = \frac{A(s-1)+B(s+1)}{(s+1)(s-1)} \\ &= \frac{(A+B)s - A + B}{(s+1)(s-1)} \end{aligned}$$

We have $\begin{cases} A+B &= 1 \\ -A+B &= 0 \end{cases} \Longrightarrow \begin{cases} A &= 1/2 \\ B &= 1/2 \end{cases}$

Therefore $F(s) = \frac{1/2}{s+1} + \frac{1/2}{s-1} = \frac{1}{2}\frac{1}{s+1} + \frac{1}{2}\frac{1}{s-1}$.

and

$y(t) = L^{-1}\{F(s)\} = \frac{1}{2}L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{s-1}\right\} = \frac{1}{2}e^{-t} + \frac{1}{2}e^{t} = \cosh t.$

16. $y'' - 2y' + 2y = e^{-t} \quad y(0) = 0, y'(0) = 1.$

Solution:

$L\left(y'' - 2y' + 2y\right) = L(e^{-t}) = \frac{1}{s+1}$.

Let $F(s) = L(y(t))$, we have $L(y'') = s^2L(y) - sy(0) - y'(0) = s^2F(s) - 1$ and $L(y') = sL(y) - y(0) = sF(s)$.

$$
\begin{aligned}
s^2F(s) - 1 - 2(sF(s)) + 2F(s) = \frac{1}{s+1} &\Longrightarrow (s^2 - 2s + 2)F(s) = \frac{1}{s+1} + 1 \\
&\Longrightarrow F(s) = \frac{\frac{1}{s+1} + 1}{s^2 - 2s + 2}.
\end{aligned}
$$

$$
\begin{aligned}
F(s) &= \frac{\frac{1}{s+1} + 1}{s^2 - 2s + 2} = \frac{s+2}{(s+1)(s^2 - 2s + 2)} \\
&= \frac{A}{s+1} + \frac{Bs + C}{s^2 - 2s + 2} \\
&= \frac{As^2 - 2As + 2A + Bs^2 + (B+C)s + c}{(s+1)(s^2 - 2s + 2)}
\end{aligned}
$$

We have $\begin{cases} A+B &= 0 \\ B+C-2A &= 1 \\ 2A+C &= 2 \end{cases} \Longrightarrow \begin{cases} A &= 1/5 \\ B &= -1/5 \\ C &= 7/5 \end{cases}$

Therefore $F(s) = \frac{1/5}{s+1} - 1/5\frac{s-1}{(s-1)^2+1} + 7/5\frac{1}{(s-1)^2+1}$.

and

$y(t) = L^{-1}\{F(s)\} = \frac{1}{5}L^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{5}L^{-1}\left\{\frac{s-1}{(s-1)^2+1}\right\} + \frac{7}{5}L^{-1}\left\{\frac{1}{(s-1)^2+1}\right\} = \frac{1}{5}e^{-t} - \frac{1}{5}e^{t}\cos t + \frac{7}{5}e^{t}\sin t.$

Section 6.3 5(b) $f(t) = \begin{cases} 0 & 0 \le t < 3 \\ -2 & 3 \le t < 5 \\ 2 & 5 \le t < 7 \\ 1 & t \ge 7 \end{cases}$

Solution:

$f(t) = (-2)(u_3 - u_5) + 2(u_5 - u_7) + u_7.$

14. $F(s) = \frac{e^{-2s}}{s^2+s-2}$

Solution:

$\frac{1}{s^2+s-2} = \frac{1}{(s+2)(s-1)} = \frac{1}{3}\left(\frac{1}{s-1} - \frac{1}{s+2}\right).$

$L^{-1}\left\{\frac{1}{s^2+s-2}\right\} = \frac{1}{3}\left(e^t - e^{-2t}\right).$

$L^{-1}\left\{F(s)\right\} = \frac{1}{3}u_2(t)\left(e^{t-2} - e^{-2(t-2)}\right).$

Section 6.4 1(b) $y'' + y = f(t) \quad y(0) = 0, y'(0) = 1, f(t) = \begin{cases} 1 & 0 \le t < 3\pi \\ 0 & 3\pi \le t < \infty \end{cases}$

Solution:

Let $F(s) = L(y)$. We have $f(t) = u_0 - u_{3\pi}$.

$L(y'') = s^2L(y) - sy(0) - y'(0) = s^2F(s) - 1$ and

$L(f(t)) = L(u_0) - L(u_{3\pi}) = \frac{1}{s} - e^{-3\pi s}\frac{1}{s}.$

$$\begin{aligned} s^2F(s) - 1 + F(s) &= \frac{1}{s}\left(1 - e^{-3\pi s}\right) \Longrightarrow (s^2+1)F(s) \\ &= \frac{1}{s}\left(1 - e^{-3\pi s}\right) + 1 \Longrightarrow F(s) = \frac{1 - e^{-3\pi s} + s}{(s^2+1)s} \end{aligned}$$

$$\begin{aligned} \frac{1}{(s^2+1)s} &= \frac{As+B}{s^2+1} + \frac{C}{s} \\ &= \frac{As^2 + Bs + Cs^2 + C}{(s^2+1)(s)} \end{aligned}$$

We have $\begin{cases} A + C &= 0 \\ B &= 0 \\ C &= 1 \end{cases} \Longrightarrow \begin{cases} A &= -1 \\ B &= 0 \\ C &= 1 \end{cases}$

Therefore $\frac{1}{(s^2+1)s} = \frac{-s}{(s^2+1} + \frac{1}{s}$,

$$\begin{aligned} F(s) &= \frac{1 - e^{-3\pi s}}{(s^2+1)s} + \frac{1}{(s^2+1)} \\ &= (1 - e^{-3\pi s})\left(\frac{-s}{s^2+1} + \frac{1}{s}\right) + \frac{1}{s^2+1}. \end{aligned}$$

and

$L^{-1}\left(\frac{-s}{s^2+1}+\frac{1}{s}\right)=-\cos t+1$

$$
\begin{aligned}
y(t) &= L^{-1}\{F(s)\} \\
&= 1-\cos t-u_{3\pi}(1-\cos(t-3\pi))+\sin t \\
&= 1-\cos t-u_{3\pi}(1+\cos t)+\sin t.
\end{aligned}
$$