Hw3 Solution

Section 2.6

3. $(3x^2 - 2xy + 2) + (6y^2 - x^2 + 3)y' = 0$
Solution: $M(x,y) = 3x^2 - 2xy + 2,\ N(x,y) = 6y^2 - x^2 + 3$
$\frac{\partial M}{\partial y} = -2x, \quad \frac{\partial N}{\partial x} = -2x = \frac{\partial M}{\partial y} \Longrightarrow$ it is exact.

$$\begin{aligned}
\phi(x,y) &= \int M(x,y)dx = \int 3x^2 - 2xy + 2dx = x^3 - x^2y + 2x + C_1(y),\\
\phi(x,y) &= \int N(x,y)dy = \int 6y^2 - x^2 + 3dy = 2y^3 - x^2y + 3y + C_2(x),\\
&\Longrightarrow\\
C_1(y) &= 2y^3 + 3y,\\
C_2(x) &= x^3 + 2x,\\
&\Longrightarrow\\
\psi(x,y) &= x^3 - x^2y + 2x + 2y^3 + 3y.
\end{aligned}$$

Therefore the solution is

$$\psi(x,y) = C \Longrightarrow x^3 - x^2y + 2x + 2y^3 + 3y = C.$$

4. $\frac{dy}{dx} = -\frac{ax+by}{bx+cy}$
Solution: $(ax+by)dx + (bx+cy)dy = 0$
$M(x,y) = ax + by,\ N(x,y) = bx + cy$
$\frac{\partial M}{\partial y} = b, \quad \frac{\partial N}{\partial x} = b = \frac{\partial M}{\partial y} \Longrightarrow$ it is exact.

$$\begin{aligned}
\phi(x,y) &= \int M(x,y)dx = \int ax + bydx = \frac{1}{2}ax^2 + bxy + C_1(y),\\
\phi(x,y) &= \int N(x,y)dy = \int bx + cydy = bxy + \frac{1}{2}y^2 + C_2(x),\\
&\Longrightarrow\\
C_1(y) &= \frac{1}{2}y^2,\\
C_2(x) &= \frac{1}{2}x^2,\\
&\Longrightarrow\\
\psi(x,y) &= \frac{1}{2}ax^2 + bxy + \frac{1}{2}y^2.
\end{aligned}$$

Therefore the solution is

$$\psi(x,y) = C \Longrightarrow \frac{1}{2}ax^2 + bxy + \frac{1}{2}y^2 = C.$$

9. $(2x-y)+(2y-x)y'=0$
Solution: $(2x-y)dx+(2y-x)dy=0$
$M(x,y)=2x-y,\ N(x,y)=2y-x$
$\frac{\partial M}{\partial y}=-1, \quad \frac{\partial N}{\partial x}=-1=\frac{\partial M}{\partial y} \Longrightarrow$ it is exact.

$$\begin{aligned}
\phi(x,y) &= \int M(x,y)dx = \int 2x-ydx = x^2-xy+C_1(y), \\
\phi(x,y) &= \int N(x,y)dy = \int 2y-xdy = -xy+y^2+C_2(x), \\
&\Longrightarrow \\
C_1(y) &= y^2, \\
C_2(x) &= x^2, \\
&\Longrightarrow \\
\psi(x,y) &= x^2-xy+y^2.
\end{aligned}$$

Therefore the solution is

$$\psi(x,y)=C \Longrightarrow x^2-xy+y^2=C.$$

Since $y(1)=3$, we have $1-3+9=C$ and $x^2-xy+y^2-7=0$.
Therefore

$$y=\frac{x\pm\sqrt{x^2-4(x^2-7)}}{2}=\frac{x\pm\sqrt{-3x^2+28}}{2}.$$

Since $y(1)=3$, we only take "+" and $y=\frac{x+\sqrt{x^2-4(x^2-7)}}{2}=\frac{x\pm\sqrt{-3x^2+28}}{2}$. This requires that $-3x^2+28\geq 0 \Longrightarrow |x|\leq\sqrt{\frac{28}{3}}$.

11. $(xy^2+bx^2y)+((x+y)x^2)y'=0$
Solution: $M(x,y)=xy^2+bx^2y,\ N(x,y)=(x+y)x^2$
$\frac{\partial M}{\partial y}=2xy+bx^2, \quad \frac{\partial N}{\partial x}=x^2+2(x+y)x=3x^2+2xy=\frac{\partial M}{\partial y} \Longrightarrow b=3$ it is exact.

$$\begin{aligned}
\phi(x,y) &= \int M(x,y)dx = \int xy^2+3x^2ydx = \frac{1}{2}x^2y^2+x^3y+C_1(y), \\
\phi(x,y) &= \int N(x,y)dy = \int (x+y)x^2dy = x^3y+\frac{1}{2}x^2y^2+C_2(x), \\
&\Longrightarrow \\
C_1(y) &= 0, \\
C_2(x) &= 0, \\
&\Longrightarrow \\
\psi(x,y) &= \frac{1}{2}x^2y^2+x^3y.
\end{aligned}$$

Therefore the solution is

$$\psi(x,y) = C \Longrightarrow \frac{1}{2}x^2y^2 + x^3y = C.$$

14. Proof: $M(x,y) = M(x)$, $N(x,y) = N(y)$
$\frac{\partial M}{\partial y} == 0, \quad \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y} \Longrightarrow$ it is exact.

Section 2.3

1. Solution: Let $Q(t)$ be the amount of the dye in the tank at time t. We have

$$\begin{cases} \frac{dQ}{dt} &= -\frac{Q}{200}\cdot 2 \\ Q(0) &= 200\cdot 1 = 200. \end{cases}$$

$$\frac{dQ}{dt} = -\frac{Q}{100} \Longrightarrow \frac{100}{Q}dQ = -dt \Longrightarrow 100\ln Q = -t + C \Longrightarrow Q = e^{\frac{C-t}{100}}.$$

$$Q(0) = 200 \to 200 = e^{\frac{C}{100}} \to C = 100\ln 200$$

Therefore

$$Q(t) = e^{\frac{100\ln 200 - t}{100}} = 200e^{-\frac{t}{100}}.$$

We want that $Q(t) = 1\%Q(0) = 2 \to 200e^{-\frac{t}{100}} = 2 \to t = 1 - -\ln 100 \approx 460.5$.

2. Solution: Let $Q(t)$ be the amount of the dye in the tank at time t. We have

$$\begin{cases} \frac{dQ}{dt} &= 2\gamma - \frac{Q}{120}\cdot 2 \\ Q(0) &= 0. \end{cases}$$

$$\frac{dQ}{dt} + \frac{Q}{60} = 2\gamma \Longrightarrow$$

(1) $p(t) = \frac{1}{60}$, $g(t) = 2\gamma$
(2) $\mu = e^{\int p(t)dt} = e^{\int \frac{1}{60}dt} = e^{t/60}$
(3) $y(t) = \frac{\int \mu(t)g(t)dt}{\mu(t)} = \frac{\int e^{t/60}2\gamma dt + c}{e^{t/60}} = \frac{120\gamma e^{t/60}+c}{e^{t/60}} = 120\gamma + ce^{-t/60}$.
$Q(0) = 0 \Longrightarrow 120\gamma + c = 0 \Longrightarrow c = -120\gamma$. Therefore

$$Q(t) = 120\gamma - 120\gamma e^{-t/60}.$$

When $t \to \infty$, we have $Q(t) \to 120\gamma$.