Hw5 Solution

Section 3.1 7. $y'' + y' - 2y = 0 \quad y(0) = 1, y'(0) = 1$

$\boxed{\text{Solution:}}$ $a = 1,\ b = 1,\ c = -2.$

$r^2 + r - 2 = 0 \Longrightarrow r = 1$ or $r = -2$. Therefore

$$y(t) = C_1e^t + C_2e^{-2t} \quad y'(t) = C_1e^t - 2C_2e^{-2t}.$$

$$\begin{aligned} y(0) = 1 &\implies C_1 + C_2 = 1 \\ y'(0) = 1 &\implies C_1 - 2C_2 = 1 \\ &\implies \\ & \quad C_1 = 1 \quad C_2 = 0 \end{aligned}$$

$$\boxed{y(t) = e^t \quad y \to +\infty \text{ as } t \to +\infty.}$$

9. $y'' + 3y' = 0, \quad y(0) = -2, y'(0) = 3.$

$\boxed{\text{Solution:}}$ $a = 1,\ b = 3,\ c = 0.$

$r^2 + 3r = 0 \Longrightarrow r = 0$ or $r = -3$. Therefore

$$y(t) = C_1 + C_2e^{-3t} \quad y'(t) = -3C_2e^{-3t}.$$

$$\begin{aligned} y(0) = -2 &\implies C_1 + C_2 = -2 \\ y'(0) = 3 &\implies -3C_2 = 3 \\ &\implies \\ & \quad C_1 = -1 \quad C_2 = -1 \end{aligned}$$

$$\boxed{y(t) = -1 - e^{-3t} \quad y \to -1 \text{ as } t \to +\infty.}$$

12. $4y'' - y = 0, \quad y(-2) = 1, y'(-2) = -1.$

$\boxed{\text{Solution:}}$ $a = 4,\ b = 0,\ c = -1.$

$4r^2 - 1 = 0 \Longrightarrow r = 1/2$ or $r = -1/2$. Therefore

$$y(t) = C_1e^{t/2} + C_2e^{-t/2} \quad y'(t) = C_1/2e^{t/2} - C_2/2e^{-t/2}.$$

$$
\begin{aligned}
y(-2) = 1 &\implies C_1 e^{-1} + C_2 e = 1 \\
y'(-2) = -1 &\implies C_1/2e^{-1} - C_2/2e = -1 \\
&\implies \\
& \quad C_1 = -e/2 \quad C_2 = 3e^{-1}/2
\end{aligned}
$$

$$\boxed{y(t) = --e/2e^{t/2} + 3/2e^{-1}e^{-t/2} \quad y \to -\infty \text{ as } t \to +\infty.}$$

14. $y'' - y = 0, \quad y(0) = 5/4, y'(0) = -3/4.$
Solution: $a = 1,\ b = 0,\ c = -1.$
$r^2 - 1 = 0 \implies r = 1$ or $r = -1$. Therefore

$$y(t) = C_1 e^t + C_2 e^{-t} \quad y'(t) = C_1 e^t - C_2 e^{-t}.$$

$$
\begin{aligned}
y(0) = 5/4 &\implies C_1 + C_2 = 5/4 \\
y'(0) = -3/4 &\implies C_1 - C_2 = -3/4 \\
&\implies \\
& \quad C_1 = 1/4 \quad C_2 = 1
\end{aligned}
$$

$$\boxed{y(t) = e^t/4 + e^{-t}}$$

$$
\begin{aligned}
y'(t) &= e^t/4 - e^{-t} = 0 \implies e^{2t} = 4 \implies t = \ln 2 \\
y''(\ln 2) &= 1/4 \times 2 + 1/2 > 0.
\end{aligned}
$$

Therefore, its minumum value is $y(\ln 2) = 1/4 \times 2 + 1/2 = 1.$

16. $y'' - y' - 2y = 0, \quad y(0) = \alpha, y'(0) = 2.$
Solution: $a = 1,\ b = -1,\ c = -2.$
$r^2 - r - 2 = 0 \implies r = 2$ or $r = -1$. Therefore

$$y(t) = C_1 e^{2t} + C_2 e^{-t} \quad y'(t) = 2C_1 e^{2t} - C_2 e^{-t}.$$

$$
\begin{aligned}
y(0) = \alpha &\implies C_1 + C_2 = \alpha \\
y'(0) = 2 &\implies 2C_1 - C_2 = 2 \\
&\implies \\
& \quad C_1 = \frac{\alpha + 2}{3} \quad C_2 = \frac{2\alpha - 2}{3}.
\end{aligned}
$$

$\boxed{y(t) = \frac{\alpha+2}{3}e^{2t} + \frac{2\alpha-2}{3}e^{-t}}$

Since we want $y \to 0$ as $t \to +\infty$, we have

$$\frac{\alpha+2}{3} = 0 \Longrightarrow \alpha = -2.$$

Section 3.2 4. Solution:

$$\begin{aligned} y_1 &= e^t \sin t, & y_2 &= e^t \cos t \\ y_1' &= e^t \sin t + e^t \cos t & y_2' &= e^t \cos t - e^t \sin t. \end{aligned}$$

Therefore

$$\begin{aligned} W &= y_1 y_2' - y_1' y_2 = e^t \sin t \left(e^t \cos t - e^t \sin t\right) - \left(e^t \sin t + e^t \cos t\right) e^t \cos t \\ &= e^{2t}\left(\sin t \cos t - \sin^2 t - \sin t \cos t - \cos^2 t\right) = -e^{2t}. \end{aligned}$$

17. $y'' + y' - 2y = 0, \quad t_0 = 0$

Solution: $a = 1$, $b = 1$, $c = -2$.

$r^2 + r - 2 = 0 \Longrightarrow r = 1$ or $r = -2$. Therefore

$$y(t) = C_1 e^t + C_2 e^{-2t} \quad y'(t) = C_1 e^t - 2C_2 e^{-2t}.$$

For y_1

$$\begin{aligned} y_1(0) = 1 &\Longrightarrow C_1 + C_2 = 1 \\ y_1'(0) = 0 &\Longrightarrow C_1 - 2C_2 = 0 \\ &\Longrightarrow \\ & \quad C_1 = 2/3 \quad C_2 = 1/3 \end{aligned}$$

$\boxed{y_1(t) = 2/3e^t + 1/3e^{-2t}.}$

For y_2

$$\begin{aligned} y_2(0) = 0 &\Longrightarrow C_1 + C_2 = 0 \\ y_2'(0) = 1 &\Longrightarrow C_1 - 2C_2 = 1 \\ &\Longrightarrow \\ & \quad C_1 = 1/3 \quad C_2 = -1/3 \end{aligned}$$

$\boxed{y_1(t) = 1/3e^t - 1/3e^{-2t}.}$

19. Solution:

$$\begin{array}{ll} y_1 = \cos(2t), & y_2 = \sin(2t) \\ y_1' = -2\sin(2t) & y_2' = 2\cos(2t) \\ y_1'' = -4\cos(2t) & y_2' = -4\sin(2t) \end{array}$$

$$\begin{aligned} y_1'' + 4y_1 &= -4\cos(2t) + 4\cos(4t) = 0 \\ y_2'' + 4y_2 &= -4\sin(2t) + 4\sin(4t) = 0 \end{aligned}$$

Therefore y_1 and y_2 are solutions of the equation $y'' + 4y = 0$.

$$\begin{aligned} W &= y_1 y_2' - y_1' y_2 = \cos(2t) 2\cos(2t) + 2\sin(2t)\sin(2t) \\ &= 2(\cos^2(2t) + \sin^2(2t)) = 2 \neq 0. \end{aligned}$$

They constitute a fundemental set of solutions.