Hw9

Section 7.1

1. $u'' + 0.5u' + 2u = 0$

   Solution: Let $x_1 = u$, $x_2 = x_1' = u'$ and we have $x_1' = x_2$.

$$u'' + 0.5u' + 2u = x_2' + 0.5x_2 + 2x_1 = 0 \Longrightarrow x_2' = -0.5x_2 - 2x_1.$$

   Therefore

$$\begin{cases} x_1' &= x_2 \\ x_2' &= -2x_1 - 0.5x_2. \end{cases}$$

4. $u'' + 0.25u' + 4u = 2\cos(3t), \quad u(0) = 1, u'(0) = -2$

   Solution: Let $x_1 = u$, $x_2 = x_1' = u'$ and we have $x_1' = x_2$.

$$u''+0.25u'+4u = x_2'+0.25x_2+4x_1 = 2\cos(3t) \Longrightarrow x_2' = -0.25x_2-4x_1+2\cos(3t).$$

   Therefore

$$\begin{cases} x_1' &= x_2 \\ x_2' &= -0.25x_2 - 4x_1 + 2\cos(3t). \end{cases} \quad \text{with the initial condition } x_1(0) = 1 \text{ and } x_2(0) = -2.$$

Section 7.2

4. $A = \begin{pmatrix} 3-2i & 1+i \\ 2-i & -2+3i \end{pmatrix}$

   Solution:

   a. $A^T = \begin{pmatrix} 3-2i & 2-i \\ 1+i & -2+3i \end{pmatrix}$

   b. $\bar{A} = \begin{pmatrix} 3+2i & 1-i \\ 2+i & -2-3i \end{pmatrix}$

   c. $A^* = \begin{pmatrix} 3+2i & 2+i \\ 1+i & -2+3i \end{pmatrix}$

7 $x = \begin{pmatrix} 2 \\ 3i \\ 1-i \end{pmatrix} \quad y = \begin{pmatrix} -1+i \\ 2 \\ 3-i \end{pmatrix}$ Solution:

a.

$$
\begin{aligned}
x^T y &= (2 \;\; 3i \;\; 1-i) \begin{pmatrix} -1+i \\ 2 \\ 3-i \end{pmatrix} \\
&= 2(-1+i) + 3i \cdot 2 + +(1-i)(3-i) \\
&= -2 + 2i + 6i + 3 - 3i - i - 4i = 4i.
\end{aligned}
$$

b.

$$
\begin{aligned}
y^T y &= (-1+i \;\; 2 \;\; 3-i) \begin{pmatrix} -1+i \\ 2 \\ 3-i \end{pmatrix} \\
&= (-1+i)^2 + 2^2 + +(3-i)^2 \\
&= 1 - 1 - 2i + 4 + 9 - 6i - 1 = 12 - 8i.
\end{aligned}
$$

c.

$$
\begin{aligned}
(x, y) &= \sum_{i=1}^{3} x_i \bar{y}_i \\
&= (2(-1-i) + 3i \cdot 2 + (1-i)(3+i) \\
&= -2 - 2i + 6i + 3 - 3i + i + 1 = 2 + 2i.
\end{aligned}
$$

d.

$$
\begin{aligned}
(y, y) &= \sum_{i=1}^{3} y_i \bar{y}_i \\
&= (-1+i)(-1-i) + 2 \cdot 2 + (3-i)(3+i) \\
&= 1 + 1 + 4 + 9 + 1 = 16.
\end{aligned}
$$

Section 7.3 6 Let $c_1 x^{(1)} + c_2 x^{(2)} + c_3 x^{(3)} = 0$, then

$$
\begin{bmatrix} c_1 \\ c_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 \\ c_2 \end{bmatrix} + \begin{bmatrix} c_3 \\ 0 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
$$

We have $\begin{cases} c_1 + c_3 = 0 \\ c_1 + c_2 = 0 \\ c_2 + c_3 = 0. \end{cases}$

Therefore $c_1 = c_2 = c_3 = 0$ and $x^{(1)}$, $x^{(2)}$, and $x^{(3)}$ are linearly independent.

8 Let $c_1 x^{(1)} + c_2 x^{(2)} + c_3 x^{(3)} + c_4 x^{(4)} = 0$, then

$$\begin{bmatrix} c_1 \\ 2c_1 \\ -c_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2c_2 \\ 3c_2 \\ c_2 \\ -c_2 \end{bmatrix} + \begin{bmatrix} -c_3 \\ 0 \\ 2c_3 \\ 2c_3 \end{bmatrix} + \begin{bmatrix} 3c_4 \\ -c_4 \\ c_4 \\ 3c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

We have $\begin{cases} c_1 + 2c_2 - c_3 + 3c_4 & = 0 \\ 2c_1 + 3c_2 - c_4 & = 0 \\ -c_1 + c_2 + 2c_3 + c_4 & = 0 \\ -c_2 + 2c_3 + 3c_4 & = 0. \end{cases}$

Therefore $c_1 = c_2 = c_3 = c_4 = 0$ and $x^{(1)}$, $x^{(2)}$, $x^{(3}$, and $x^{(4)}$ are linearly independent.

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$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{vmatrix} = (\lambda - 5)(\lambda - 1) + 3 = \lambda^2 - 6\lambda + 8 = 0 \rightarrow \lambda_1 = 2, \lambda_2 = 4.$$

For $\lambda_1 = 2$

$$\begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Longrightarrow 5x_1 - x_2 = 2x_1 \Longrightarrow x_2 = 3x_1.$$

The eigenvector is $\xi_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

For $\lambda_2 = 4$

$$\begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Longrightarrow 5x_1 - x_2 = 4x_1 \Longrightarrow x_2 = x_1.$$

The eigenvector is $\xi_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

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$$|A-\lambda I| = \begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = (\lambda+2)(\lambda+2)-1 = \lambda^2+4\lambda+3 = 0 \to \lambda_1 = -1, \lambda_2 = -3.$$

For $\lambda_1 = -1$

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Longrightarrow -2x_1 + x_2 = -x_1 \Longrightarrow x_2 = x_1.$$

The eigenvector is $\xi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For $\lambda_2 = -3$

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -3 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Longrightarrow -2x_1 + x_2 = -3x_1 \Longrightarrow x_2 = -x_1.$$

The eigenvector is $\xi_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.