Name:____________________ **Total:**____________________

Math 320 Midterm Exam II (11/18/2019)

1. (20 points)

Consider the differential equation

$$9y'' + 24y' + 16y = 0.$$

Find two fundamental solutions and calculate the Wronskian of these two solutions.

Solution. $9r^2 + 24r + 16 = 0$ 2

$(3r+4)^2 = 0 \Rightarrow r = -\frac{4}{3}$ 2

$y_1 = e^{-\frac{4}{3}t}$ 3 $\qquad y_2 = te^{-\frac{4}{3}t}$ 3

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2$$ 5

$$= e^{-\frac{4}{3}t}\left(te^{-\frac{4}{3}t}\right)' - \left(e^{-\frac{4}{3}t}\right)' te^{-\frac{4}{3}t}$$

$$= e^{-\frac{4}{3}t}\left(e^{-\frac{4}{3}t} - \frac{4}{3}te^{-\frac{4}{3}t}\right) + \frac{4}{3}e^{-\frac{4}{3}t}te^{-\frac{4}{3}t}$$

$$= e^{-\frac{8t}{3}}\left(1 - \cancel{\frac{4}{3}t} + \cancel{\frac{3}{4}t}\right)$$

$$= e^{-\frac{8t}{3}}$$ 5

Therefore y_1, y_2 form a fundamental set of solutions,

2. (20 points)

Find the solution of the following equation:

$$y'' + y' - 6y = e^{2t}.$$

Solution: $r^2 + r - 6 = 0$ (2) $\quad (r-2)(r+3) = 0 \Rightarrow r_1 = 2, \quad r_2 = -3$ (2)

$y_1 = e^{2t}$ (2) $\quad y_2 = e^{-3t}$ (2)

$Y = tAe^{2t}$ (3) $\quad Y' = Ae^{2t} + 2Ate^{2t}$ (1)

$Y'' = 2Ae^{2t} + 2Ae^{2t} + 4Ate^{2t} = 4Ae^{2t} + 4Ate^{2t}$ (1)

$Y'' + Y' - 6Y = e^{2t}$ (1)

$4Ae^{2t} + \cancel{4Ate^{2t}} + Ae^{2t} + \cancel{2Ate^{2t}} - \cancel{6teAe^{2t}} = e^{2t}$

$5Ae^{2t} + 0 = e^{2t}$

$\Rightarrow 5A = 1 \Rightarrow A = \frac{1}{5}$ (3)

$Y = \frac{1}{5}te^{2t}$

$y(t) = c_1y_1 + c_2y_2 + Y = c_1e^{2t} + c_2e^{-3t} + \frac{1}{5}te^{2t}$ (3)

3. (20 points) Find the general solution of the given system of equations in term of real-valued functions.

$$\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \mathbf{x}.$$

Solution $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$

$$\det(A - rI) = \det \begin{pmatrix} 3-r & -2 \\ 4 & -1-r \end{pmatrix} = (r-3)(r+1)+8 = r^2 - 2r + 5 = 0$$

2

$$r = \frac{2 \pm \sqrt{4-4\cdot 5}}{2} = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i$$

2

for $r = 1+2i$ eigenvector $\xi = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (1+2i) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{aligned} 3x_1 - 2x_2 &= (1+2i)x_1 \\ 4x_1 - x_2 &= (1+2i)x_2 \end{aligned}$$

3

$$\Rightarrow \quad 2x_2 = (2-2i)x_1 \Rightarrow x_2 = (1-i)x_1$$

$$(2+2i)x_2 = 4x_1 \Rightarrow x_2 = \frac{2}{1+i} x_1 = (1-i)x_1$$

take $x_1 = 1$, $x_2 = 1-i$ $\quad \xi = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} i$

3 2 2

$\lambda = 1, \mu = 2, \quad a = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

1

$$y(t) = c_1 e^{\lambda t}(a \cos\mu t - b \sin\mu t) + c_2 e^{\lambda t}(a \sin\mu t + b\cos\mu t)$$

$$= c_1 e^t \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos 2t - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin 2t \right] + c_2 e^{t} \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin 2t + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos 2t \right]$$

2 2

4. (20 points) Find the solution of the given system of equations with the initial conditions.

$$\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Solution, $A = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}$

$$\det(A - rI) = \det \begin{pmatrix} 1-r & -4 \\ 4 & -7-r \end{pmatrix} = (r-1)(r+7)+16 = r^2+6r+9 = (r+3)^2 = 0$$

$\Rightarrow r = -3$

3

for $r=-3$, the corresponding eigenvector $\xi = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -3\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{matrix} x_1 - 4x_2 = -3x_1 \\ 4x_1 - 7x_2 = -3x_2 \end{matrix} \Rightarrow \begin{matrix} x_1 = x_2 \\ x_1 = x_2 \end{matrix}$$

take $x_1 = 1 \Rightarrow x_2 = 1$

$\xi = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

5

$$(A - rI)\eta = \xi \Rightarrow \begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \begin{matrix} 4x_1 - 4x_2 = 1 \\ 4x_1 - 4x_2 = 1 \end{matrix} \Rightarrow x_1 = \frac{1+4x_2}{4}$$

take $x_2 = 0$, $x_1 = \frac{1}{4}$

5

$\eta = \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}$

$$x(t) = C_1 \xi e^{rt} + C_2(\xi t e^{rt} + \eta e^{rt})$$

$$= C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} + C_2 \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-3t} + \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} e^{-3t} \right)$$

2

$$x(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \Rightarrow C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \Rightarrow \begin{matrix} C_1 + \frac{1}{4}C_2 = 3 \\ C_1 = 2 \end{matrix} \Rightarrow \begin{matrix} C_1 = 2 \\ C_2 = 4 \end{matrix}$$

3

$$x(t) = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} + 4\begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-3t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-3t}$$

$$= \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-3t} + \begin{pmatrix} 4 \\ 4 \end{pmatrix} t e^{-3t}$$

2

5. (20 points + 10 points) A mass weighing 3lb stretches a spring 6 in. If the mass is pulled downward, stretching the spring a distance of 3 in, and set in motion with a downward velocity of 2ft/s, and there is no damping, determine the position u of the mass at any time t. Determine the frequency, period, amplitude, and phase of the motion.

When does the mass first return to its equilibrium position?(Bonus question with additional 10 points).

Solution:

$W = 3lb$ $\qquad$ $u(0) = 3 \text{ in} = \frac{3}{12} = \frac{1}{4} ft$

$L = 6in = \frac{6}{12} ft = \frac{1}{2} ft$ $\qquad$ $u'(0) = 2$

$m = \frac{W}{g} = \frac{3}{32}$ (1)

$mg = kL \Rightarrow k = \frac{mg}{L} = \frac{3}{\frac{1}{2}} = 6$ (1)

$mu'' + ku = 0 \Rightarrow \begin{cases} \frac{3}{32}u'' + 6u = 0 \\ u(0) = \frac{1}{4} \\ u'(0) = 2 \end{cases}$ (3)

$\frac{3}{32}r^2 + 6 = 0 \Rightarrow r^2 = -64 \quad r = \pm 8i$

$u(t) = C_1 \cos 8t + C_2 \sin 8t \qquad u' = -8C_1 \sin 8t + 8C_2 \cos 8t$ (5)

$u(0) = C_1 = \frac{1}{4} \Rightarrow C_1 = \frac{1}{4}$

$u'(0) = 8C_2 = 2 \Rightarrow C_2 = \frac{1}{4}$ (2)

$u(t) = \frac{1}{4}\cos 8t + \frac{1}{4}\sin 8t = \frac{\sqrt{2}}{4}\cos\left(8t - \frac{\pi}{4}\right)$

$R = \sqrt{\frac{1}{4}^2 + \frac{1}{4}^2} = \frac{\sqrt{2}}{4}$ $\qquad$ $\tan\delta = 1 \Rightarrow \delta = \frac{\pi}{4}$

frequency $\omega = 8$ (2)

period $T = \frac{2\pi}{\omega} = \frac{\pi}{4}$ (2)

amplitude $R = \frac{\sqrt{2}}{4}$ (2)

phase $\delta = \frac{\pi}{4}$ (2)

when the mass return to its equilibrium $u(t) = 0$

$\frac{\sqrt{2}}{4}\cos\left(8t - \frac{\pi}{4}\right) = 0$ $\qquad$ $8t - \frac{\pi}{4} = \frac{\pi}{2}$

$\Rightarrow t = \frac{3\pi}{32}$

≈ 0.2949