Math 781 Hw10 Solution

1. Derive the formula for approximating the derivative.

$$f'(x) \approx \frac{1}{2h}\left(-3f(x) + 4f(x+h) - f(x+2h)\right).$$

Solution: We can use the method of underdetermined coefficients: assume

$$f'(x) \approx Af(x) + Bf(x+h) + Cf(x+2h).$$

Using the Taylor's formulas of $f(x), f(x+h), f(x+2h)$, we have

$$\begin{aligned}
f(x) &= f(x) \\
f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + \cdots \\
f(x+2h) &= f(x) + 2hf'(x) + \frac{4h^2}{2}f''(x) + \frac{8h^3}{6}f'''(x) + \cdots
\end{aligned}$$

Then

$$\begin{aligned}
& Af(x) + Bf(x+h) + Cf(x+2h) \\
= \; & (A+B+C)f(x) + h(B+2C)f'(x) + \frac{h^2}{2}(B+4C)f''(x) + \frac{h^3}{6}(B+8C)f'''(x)) + \cdots.
\end{aligned}$$

We require

$$A + B + C = 0, \quad B + 2C = \frac{1}{h}, \quad B + 4C = 0,$$

and obtain

$$A = -\frac{3}{2h}, \quad B = \frac{4}{2h}, \quad C = -\frac{1}{2h}.$$

Therefore

$$f'(x) \approx \frac{1}{2h}\left(-3f(x) + 4f(x+h) - f(x+2h)\right).$$

2. Using Taylor series to derive the error term for the formula in Problem 1.

Solution: From Problem 1, we have the error term is given by

$$\frac{h^3}{6}(B+8C)f'''(x)) + \cdots = -\frac{1}{3}h^2 f'''(x) + \cdots = -\frac{1}{3}h^2 f'''(\xi).$$

3. Suppose that $N(h)$ is an approximation to M for every $h > 0$ and

$$M - N(h) = K_1 h + K_2 h^2 + K_3 h^3 + \cdots,$$

where K_1, K_2, K_3 are nonzeros constants independent of h. Use $N(h)$, $N(h/2)$, and $N(h/3)$ to produce an $O(h^3)$ approximation to M.

Solution: Since

$$\begin{aligned} M - N(h) &= K_1 h + K_2 h^2 + K_3 h^3 + \cdots \\ M - N(\frac{h}{2}) &= K_1 \frac{h}{2} + K_2 \frac{h^2}{4} + K_3 \frac{h^3}{8} + \cdots \\ M - N(\frac{h}{3}) &= K_1 \frac{h}{3} + K_2 \frac{h^2}{9} + K_3 \frac{h^3}{27} + \cdots \end{aligned}$$

The second equation multiply 2, minus the first equation and the third equation multiply 3, minus the first equation. We have

$$\begin{aligned} M - 2N(\frac{h}{2}) + N(h) &= -\frac{1}{2} K_2 h^2 - \frac{3}{4} K_3 h^3 + \cdots \\ 2M - 3N(\frac{h}{3}) + N(h) &= -\frac{2}{3} K_2 h^2 - \frac{8}{9} K_3 h^3 + \cdots \end{aligned}$$

The above first equation multiply 4 and minus the second equation multiply by 3, we have

$$-2M - 8N(\frac{h}{2}) + 4N(h) + 9N(\frac{h}{3}) - 3N(h) = -3K_3 h^3 + \frac{8}{3} K_3 h^3 + \cdots$$

Therefore

$$-2M - 8N(\frac{h}{2}) + 4N(h) + 9N(\frac{h}{3}) - 3N(h) = -\frac{1}{3} K_3 h^3 + \cdots$$

and

$$M = \frac{1}{2} N(h) - 4N(\frac{h}{2}) + \frac{9}{2} N(\frac{h}{3}) + O(h^3).$$

4. Derive a numerical differentiation formula of order $O(h^4)$ by applying Richardson extrapolation to

$$f'(x) = \frac{1}{2h} \left(f(x+h) - f(x-h) \right) - \frac{h^2}{6} f'''(x) - \frac{h^4}{120} f^{(5)}(x) + \cdots$$

Give the error term of order $O(h^4)$.

Solution: Let $M = f'(x),\ N(h) = \frac{1}{2h}\left(f(x+h) - f(x-h)\right),\ K_1 = -\frac{1}{6}f'''(x),\ K_2 = -\frac{1}{120}f^{(5)}(x)$, we have

$$\begin{aligned} M - N(h) &= K_1 h^2 + K_2 h^4 + \cdots \\ M - N(\frac{h}{2}) &= K_1 \frac{h^2}{4} + K_2 \frac{h^4}{16} + \cdots \end{aligned}$$

The second equation multiply 4, minus the first equation. We have

$$3M - 4N(\frac{h}{2}) + N(h) = -\frac{3}{4}K_2 h^4 + \cdots$$

Therefore

$$M = \frac{4}{3}N(\frac{h}{2}) - \frac{1}{3}N(h) - \frac{1}{4}K_2 h^4 + \cdots = \frac{4}{3}N(\frac{h}{2}) - \frac{1}{3}N(h) + \frac{1}{480}f^{(5)}(x)h^4 + \cdots$$