Math 781 Hw11 Solution

1. Verify the following formula is exact for polynomials of degree ≤ 4.

$$\int_0^1 f(x)dx \approx \frac{1}{90}\left[7f(0) + 32f\left(\frac{1}{4}\right) + 12f\left(\frac{1}{2}\right) + 32f\left(\frac{3}{4}\right) + 7f(1)\right].$$

Proof.

For $f(x) = 1$,

$$\int_0^1 f(x)dx = \int_0^1 1dx = 1$$
$$\frac{1}{90}\left[7f(0) + 32f\left(\frac{1}{4}\right) + 12f\left(\frac{1}{2}\right) + 32f\left(\frac{3}{4}\right) + 7f(1)\right] = \frac{1}{90}[7 + 32 + 12 + 32 + 7] = 1.$$

For $f(x) = x$,

$$\int_0^1 f(x)dx = \int_0^1 xdx = \frac{1}{2}$$
$$\frac{1}{90}\left[7f(0) + 32f\left(\frac{1}{4}\right) + 12f\left(\frac{1}{2}\right) + 32f\left(\frac{3}{4}\right) + 7f(1)\right] = \frac{1}{90}\left[32\frac{1}{4} + 12\frac{1}{2} + 32\frac{3}{4} + 7\right] = \frac{1}{2}.$$

For $f(x) = x^2$,

$$\int_0^1 f(x)dx = \int_0^1 x^2dx = \frac{1}{3}$$
$$\frac{1}{90}\left[7f(0) + 32f\left(\frac{1}{4}\right) + 12f\left(\frac{1}{2}\right) + 32f\left(\frac{3}{4}\right) + 7f(1)\right] = \frac{1}{90}\left[32\frac{1}{16} + 12\frac{1}{4} + 32\frac{9}{16} + 7\right] = \frac{1}{3}.$$

For $f(x) = x^3$,

$$\int_0^1 f(x)dx = \int_0^1 x^3dx = \frac{1}{4}$$
$$\frac{1}{90}\left[7f(0) + 32f\left(\frac{1}{4}\right) + 12f\left(\frac{1}{2}\right) + 32f\left(\frac{3}{4}\right) + 7f(1)\right] = \frac{1}{90}\left[32\frac{1}{64} + 12\frac{1}{8} + 32\frac{27}{64} + 7\right] = \frac{1}{4}.$$

For $f(x) = x^4$,

$$\int_0^1 f(x)dx = \int_0^1 x^4dx = \frac{1}{5}$$
$$\frac{1}{90}\left[7f(0) + 32f\left(\frac{1}{4}\right) + 12f\left(\frac{1}{2}\right) + 32f\left(\frac{3}{4}\right) + 7f(1)\right] = \frac{1}{90}\left[32\frac{1}{256} + 12\frac{1}{16} + 32\frac{81}{256} + 7\right] = \frac{1}{5}.$$

Therefore, the formula is exact for polynomials of degree ≤ 4.

2. Find the formula

$$\int_0^1 f(x)dx \approx A_0 f(0) + A_1 f(1)$$

that is exact for all functions of the form $f(x) = ae^x + b\cos(\pi x/2)$.

Solution:

$$\begin{aligned}\int_0^1 f(x)dx &= \int_0^1 ae^x + b\cos(\pi x/2)dx = \left(ae^x + b\frac{2}{\pi}\sin(\pi x/2)\right)|_0^1 = a(e-1) + b\frac{2}{\pi} \\ &= A_0 f(0) + A_1 f(1) = A_0(a+b) + A_1 ae = (A_0 + A_1 e)a + A_0 b.\end{aligned}$$

Therefore, we have $A_0 + A_1 e = e - 1$ and $A_0 = \frac{2}{\pi}$. These two equations give us $A_0 = \frac{2}{\pi}$ and $A_1 = 1 - \frac{1+\frac{2}{\pi}}{e}$.

3. Use the Lagrange interpolation polynomial to derive the formula of the form

$$\int_0^1 f(x)dx \approx Af\left(\frac{1}{3}\right) + Bf\left(\frac{2}{3}\right).$$

Transform this formula to one for integration over $[a, b]$.

Solution: : Given two points $\frac{1}{3}$ and $\frac{2}{3}$, the interpolation polynomial

$$P_1 = f\left(\frac{1}{3}\right)L_0(x) + f\left(\frac{2}{3}\right)L_1(x),$$

where $L_0(x) = \frac{x-\frac{2}{3}}{\frac{1}{3}-\frac{2}{3}}$ and $L_1(x) = \frac{x-\frac{1}{3}}{\frac{2}{3}-\frac{1}{3}}$. Therefore, we have

$$\begin{aligned}A &= \int_0^1 L_0(x)dx = \int_0^1 \frac{x-\frac{2}{3}}{\frac{1}{3}-\frac{2}{3}}dx = -\frac{3}{2}\left(\frac{1}{9} - \frac{4}{9}\right) = \frac{1}{2} \\ B &= \int_0^1 L_1(x)dx = \int_0^1 \frac{x-\frac{1}{3}}{\frac{2}{3}-\frac{1}{3}}dx = \frac{3}{2}\left(\frac{4}{9} - \frac{1}{9}\right) = \frac{1}{2}.\end{aligned}$$

The formula is

$$\int_0^1 f(x)dx \approx \frac{1}{2}f\left(\frac{1}{3}\right) + \frac{1}{2}f\left(\frac{2}{3}\right).$$

Let $x = (b-a)t + a$, $dx = (b-a)dt$, and $t = \frac{1}{b-a}x - \frac{a}{b-a}$, we have

$$\begin{aligned}\int_a^b f(x)dx &= \int_0^1 f((b-a)t + a)(b-a)dt \\ &= (b-a)\left(\frac{1}{2}f\left(\frac{b+2a}{3}\right) + \frac{1}{2}f\left(\frac{2b+a}{3}\right)\right).\end{aligned}$$

4. Determine values for A, B, C that make the formula

$$\int_0^2 xf(x)dx \approx Af(0) + Bf(1) + Cf(2)$$

exact for all polynomials of degree as high as possible. What is the maximum degree?

Solution: :

For $f(x) = 1$,

$$\int_0^2 xf(x)dx = \int_0^2 1dx = 2$$
$$Af(0) + Bf(1) + Cf(2) = A + B + C = 2.$$

For $f(x) = x$,

$$\int_0^2 xf(x)dx = \int_0^2 x^2dx = \frac{8}{3}$$
$$Af(0) + Bf(1) + Cf(2) = A \cdot 0 + B \cdot 1 + C \cdot 2 = \frac{8}{3}.$$

For $f(x) = x^2$,

$$\int_0^2 xf(x)dx = \int_0^2 x^3dx = \frac{16}{4} = 4$$
$$Af(0) + Bf(1) + Cf(2) = A \cdot 0 + B \cdot 1 + C \cdot 4 = 4.$$

Solving these three equations and obtain $A = 0$, $B = \frac{4}{3}$, and $C = \frac{2}{3}$.

For $f(x) = x^3$,

$$\int_0^2 xf(x)dx = \int_0^2 x^4dx = \frac{32}{5}$$
$$Af(0) + Bf(1) + Cf(2) = A \cdot 0 + B \cdot 1 + C \cdot 8 = \frac{20}{3} \neq \frac{32}{5}.$$

Therefore, 2 is the maximum degree of polynomials which the formula gives the exact integral.