Math 781 Hw12

due Monday 11/21/2022.

1. Find a formula of the form
$$\int_0^1 xf(x)dx \approx \sum_{i=0}^{n} A_i f(x_i)$$
with $n = 1$, that is exact for all polynomials of degree 3. (If it is too hard to solve the nonlinear system, you can just write down the system and get full points.)

2. Using only $f(0)$, $f'(-1)$, and $f''(1)$, compute an approximation to $\int_{-1}^{1} f(x)dx$ that is exact for all quadratic polynomials. Is the approximation exact for polynomials of degree 3? Why or why not?

3. Show how the Guassian quadrature rule
$$\int_{-1}^{1} f(x)dx \approx \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$
can be used for $\int_a^b f(x)dx$. Apply this result to evaluate $\int_0^{\pi/2} xdx$.

4. Consider the quadrature
$$I_n(f) = w_1 f(x_1) + \cdots + w_n f(x_n)$$
for approximation $I(f) = \int_a^b f(x)dx$. Prove that for any choices of distinct real numbers $x_1, \cdots, x_n \in [a, b]$ and real numbers $w_1, \cdots, w_n$, the degree of precision of $I_n(f)$ is at most $2n - 1$. (Hint: Use $P(x) = \Pi_{j=1}^{n}(x - x_j)^2$ to show that $I(P) \neq I_n(P)$.)

5. Suppose that $S(f, h)$ is a quadrature rule for the integral $I(f) = \int_a^b f(x)dx$ and that the error series is
$$I(f) = S(f, h) + C_4 h^4 + C_6 h^6 + \cdots.$$
Combine $S(f, h)$ and $S(f, \frac{h}{3})$ to find a more accurate approximation of $I(f)$.