Math 781 Hw1 Solution

1. The correct assertion is "d".

2. The corrections are

   a. $e^x - 1 = O(x)$ as $x \to 0$

   b. $\cot x = O(x)$ as $x \to 0$

   c. $\cot x = o(1)$ as $x \to 0$

3. Show that if $x_n = O(\alpha_n)$, then $x_n/\ln n = o(\alpha_n)$.

   Proof. Since $x_n = O(\alpha_n)$, there exist $C > 0$ and $N_1 > 0$ such that

   $$|x_n| \le C|\alpha_n|, \quad \forall n > N_1.$$

   Since $\lim_{n\to+\infty} \frac{1}{\ln n} = 0$, $\left|\frac{x_n}{\alpha_n \ln n}\right| \le \frac{C}{\ln n} \to 0$ as $n \to \infty$. Therefore, $x_n/\ln n = o(\alpha_n)$.

4. Define a sequence by

   $$x_0 = 2; \quad x_{n+1} = \frac{1}{2}x_n + \frac{1}{x_n} \quad \forall\ n > 0.$$

   It is known that $x_n \to \sqrt{2}$. Prove that the order of the convergence is quadratic.

   Proof.

   $$\begin{aligned} \frac{|x_{n+1} - \sqrt{2}|}{|x_n - \sqrt{2}|^2} &= \frac{|\frac{1}{2}x_n + \frac{1}{x_n} - \sqrt{2}|}{|x_n - \sqrt{2}|^2} = \frac{|x_n^2 + 2 - 2\sqrt{2}x_n|}{2|x_n||x_n - \sqrt{2}|^2} \\ &= \frac{|x_n - \sqrt{2}|^2}{2|x_n||x_n - \sqrt{2}|^2} = \frac{1}{2|x_n|} \to \frac{1}{2\sqrt{2}}. \end{aligned}$$

   Therefore, $\{x_n\}$ converges to $\sqrt{2}$ quadratically.

5. Convert $x = 12.5$ into a binary expression.

   Solution. $\frac{\ln x}{\ln 2} < m \le \frac{\ln x}{\ln 2} + 1$. Since $\frac{\ln x}{\ln 2} = \frac{\ln 12.5}{\ln 2} = 3.6439 \to m = 4$.

   $$\begin{aligned} x_1 &= x2^{-m} = 12.5 \cdot 2^{-4} = 0.7812 \\ a_1 &= [2x_1] = [1.5625] = 1 \\ x_2 &= 2x_1 - a_1 = 0.5625 \\ a_2 &= [2x_2] = [1.1250] = 1 \end{aligned}$$

$$
\begin{aligned}
x_3 &= 2x_2 - a_2 = 0.1250 \\
a_3 &= [2x_3] = [0.25] = 0 \\
x_4 &= 2x_3 - a_3 = 0.25 \\
a_4 &= [2x_4] = [0.5] = 0 \\
x_5 &= 2x_4 - a_4 = 0.5 \\
a_5 &= [2x_5] = 1.
\end{aligned}
$$

$x = 12.5 = (0.11001)_2 \cdot 2^4.$