Math 781 Hw3 Solution

1. How many bits of precision are lost in a computer when we carry out the subtraction $x - \sin x$ for $x = \frac{1}{2}$?

   Solution: Since $x = x$ and $y = \sin x$, $1 - \frac{y}{x} = 1 - \frac{\sin x}{x}$.

   For $x = \frac{1}{2}$,

   $$2^{-5} \leq 1 - \frac{\sin \frac{1}{2}}{\frac{1}{2}} = 0.0411 \leq 2^{-4}.$$

   Therefore, at most 5 and at least 4 bits of precision are lost.

2. Suggest ways to avoid loss of significance in these calculations.

   (a) $\sqrt{x^2+1} - x$

   (b) $x^{-3}(\sin x - x)$

   (c) $\sqrt{x+2} - \sqrt{x}$

   Solution:

   (a)

   $$\sqrt{x^2+1} - x = \frac{\left(\sqrt{x^2+1} - x\right)\left(\sqrt{x^2+1} + x\right)}{\sqrt{x^2+1} + x} = \frac{x^2+1-x^2}{\sqrt{x^2+1}+x} = \frac{1}{\sqrt{x^2+1}+x}.$$

   (b)

   $$x^{-3}(\sin x - x) = x^{-3}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots - x\right) = -\frac{1}{3!} + \frac{x^2}{5!} - \cdots.$$

   (c)

   $$\sqrt{x+2} - \sqrt{x} = \frac{\left(\sqrt{x+2} - \sqrt{x}\right)\left(\sqrt{x+2} + \sqrt{x}\right)}{\sqrt{x+2} + \sqrt{x}} = \frac{2}{\sqrt{x+2} + \sqrt{x}}.$$

3. Find analytically the solution of this difference equation with the given initial values:

   $$\begin{cases} x_0 = 1, \quad x_1 = 0.9 \\ x_{n+1} = -0.2x_n + 0.99x_{n-1}. \end{cases}$$

   Without computing the solution recursively, predict whether such a computation would be stable.

Solution: Let $x_n = r^n$, $x_{n+1} = -0.2x_n + 0.99x_{n-1}$ gives $r^{n+1} = 0.2r^n + 0.99r^{n-1} \to r^2 + 0.2r - 0.99 = 0 \to r_1 = 0.9, r_2 = -1.1$. Therefore, $x_n = C_1 0.9^n + C_2(-1.1)^n$. The initial conditions $x_0 = 1$ and $x_1 = 0.99$ gives $C_1 + C_2 = 1$ and $C_1 0.9 + C_2(-1.1) = 0.9$. We have $C_1 = 1$ and $C_2 = 0$. So $x_n = 0.9^n$.

Since $(-1.1)^n \to +\infty$ as $n \to +\infty$, such a computation would be unstable.

4. What are the condition numbers of the following functions? Where are they large?

$$(1) f(x) = (x-1)^\alpha; \quad (2) \ln x; \quad (3) \sin x.$$

Solution:

(1) $f(x) = (x-1)^\alpha$, $f'(x) = \alpha(x-1)^{\alpha-1}$, therefore, the condition number is

$$\frac{xf'(x)}{f(x)} = \frac{x\alpha(x-1)^{\alpha-1}}{(x-1)^\alpha} = \frac{\alpha x}{x-1}.$$

When x close to 1, the condition number is large.

(2) $f(x) = \ln x$, $f'(x) = \frac{1}{x}$, therefore, the condition number is

$$\frac{xf'(x)}{f(x)} = \frac{x\frac{1}{x}}{\ln x} = \frac{1}{\ln x}.$$

When x close to 1, the condition number is large.

(3) $f(x) = \sin x$, $f'(x) = \cos x$, therefore, the condition number is

$$\frac{xf'(x)}{f(x)} = \frac{x\cos x}{\sin x}.$$

When x close to $\pm k\pi$, $k = 1, 2, \cdots$, the condition number is large.