Math 781 Hw4

due Monday 09/19/2022.

1. Consider the bisection method starting with the interval $[1.5, 3.5]$.
   (a) What is the width of the interval at the nth step of this method?
   (b) What is the maximum distance possible between the root r and the midpoint of this interval?

2. Let the bisection method be applied to a continuous function, resulting in intervals $[a_1, b_1]$, and so on. $c_n = \frac{a_n+b_n}{2}$. Let $r = \lim_{n\to\infty} a_n$. Which of the these statements can be false?
   (a) $a_1 \le a_2 \le a_3 \le \cdots$
   (b) $|r - (a_n + b_n)/2| \le 2^{-n}(b_1 - a_1) \quad n \ge 1$
   (c) $|r - (a_{n+1} + b_{n+1})/2| \le |r - (a_n + b_n)/2|$
   (d) $]a_{n+1}, b_{n+1}] \subseteq [a_n, b_n] \quad n \ge 1$
   (e) $|r - a_n| = O(2^{-n})$ as $n \to \infty$
   (f) $|r - c_n| < |r - c_{n-1}| \quad n \ge 2$

3. If the bisection method is used starting with the interval $[2, 3]$, how many steps must be taken to compute a root with absolute accuracy $< 10^{-6}$?

4. Suppose the sequence $\{p_n\}$ converges to p and there is a constant $0 \le k < 1$ such that

$$|p_n - p| \le k|p_{n-1} - p|, \quad \forall n \ge 1.$$

   Prove that

$$|p_n - p| \le \frac{k}{1-k}|p_n - p_{n-1}|, \quad \forall n \ge 1.$$

5. Consider the fixed-point iteration

$$p_n = \frac{p_{n-1}}{2} + \frac{1}{p_{n-1}}, \quad n = 1, 2, \cdots.$$

   (a) Determine the function $g(x)$ used in the iteration.
   (b) Find the fixed point(s) of $g(x)$.
   (c) Show that $\lim_{n\to\infty} p_n = \sqrt{2}$ for any $p_0 > \sqrt{2}$. (Hint: Show $\sqrt{2} < p_n < p_{n-1}$ by induction. You may need the inequalities (1) $a + b > 2\sqrt{ab}$ for $a, b > 0$ and $a \ne b$; (2) $\frac{1}{x} < \frac{x}{2}$ for $x > \sqrt{2}$.)

(d) Use the fact that $(p_0 - \sqrt{2})^2 > 0$ whenever $p_0 \neq \sqrt{2}$ to show that if $0 < p_0 < \sqrt{2}$, then $p_1 > \sqrt{2}$. (Hint: Show $p_1 - \sqrt{2} = \frac{(p_0 - \sqrt{2})^2}{2p_0}$.)

(e) Using (c) and (d) to show $\lim_{n\to\infty} p_n = \sqrt{2}$ for any $p_0 > 0$.