Math 781 Hw4 Solution

1. Consider the bisection method starting with the interval $[1.5, 3.5]$.

   (a) What is the width of the interval at the nth step of this method?

   (b) What is the maximum distance possible between the root r and the midpoint of this interval?

   Solution: $a_1 = a = 1.5$, $b_1 = b = 3.5$

   (a)

$$b_n - a_n = \left(\frac{1}{2}\right)^{n-1}(b_1 - a_1) = \left(\frac{1}{2}\right)^{n-1}(3.5 - 1.5) = \left(\frac{1}{2}\right)^{n-2}.$$

   (b)

$$|r - c_n| \leq \frac{1}{2}(b_n - a_n) = \frac{1}{2}\left(\frac{1}{2}\right)^{n-2} = \left(\frac{1}{2}\right)^{n-1}.$$

2. Let the bisection method be applied to a continuous function, resulting in intervals $[a_1, b_1]$, and so on. $c_n = \frac{a_n+b_n}{2}$. Let $r = \lim_{n\to\infty} a_n$. Which of the these statements can be false?

   (a) $a_1 \leq a_2 \leq a_3 \leq \cdots$

   (b) $|r - (a_n + b_n)/2| \leq 2^{-n}(b_1 - a_1) \quad n \geq 1$

   (c) $|r - (a_{n+1} + b_{n+1})/2| \leq |r - (a_n + b_n)/2|$

   (d) $]a_{n+1}, b_{n+1}] \subseteq [a_n, b_n] \quad n \geq 1$

   (e) $|r - a_n| = O(2^{-n})$ as $n \to \infty$

   (f) $|r - c_n| < |r - c_{n-1}| \quad n \geq 2$

   Solution: (c) and (f).

3. If the bisection method is used starting with the interval $[2, 3]$, how many steps must be taken to compute a root with absolute accuracy $< 10^{-6}$?

   Solution: $a_1 = 2$ and $b_1 = 3$.

$$|r - c_n| \leq \left(\frac{1}{2}\right)^{n}(b_1 - a_1) \leq 10^{-6} \to n \geq \frac{\log(3-2) - \log(10^{-6})}{\log 2} = 19.9$$

   Therefore, 20 steps must be taken.

4. Suppose the sequence $\{p_n\}$ converges to p and there is a constant $0 \le k < 1$ such that

$$|p_n - p| \le k|p_{n-1} - p|, \quad \forall n \ge 1.$$

Prove that

$$|p_n - p| \le \frac{k}{1-k}|p_n - p_{n-1}|, \quad \forall n \ge 1.$$

Proof.

$$|p_n - p| \le k|p_{n-1} - p| = k|p_{n-1} - p_n + p_n - p| \le k\left(|p_{n-1} - p_n| + |p_n - p|\right)$$
$$\to$$
$$(1-k)|p_n - p| \le k|p_{n-1} - p_n|$$
$$\to$$
$$|p_n - p| \le \frac{k}{1-k}|p_{n-1} - p_n|.$$

5. Consider the fixed-point iteration

$$p_n = \frac{p_{n-1}}{2} + \frac{1}{p_{n-1}}, \quad n = 1, 2, \cdots.$$

(a) Determine the function $g(x)$ used in the iteration.

(b) Find the fixed point(s) of $g(x)$.

(c) Show that $\lim_{n\to\infty} p_n = \sqrt{2}$ for any $p_0 > \sqrt{2}$. (Hint: Show $\sqrt{2} < p_n < p_{n-1}$ by induction. You may need the inequalities (1) $a + b > 2\sqrt{ab}$ for $a, b > 0$ and $a \ne b$; (2) $\frac{1}{x} < \frac{x}{2}$ for $x > \sqrt{2}$.)

(d) Use the fact that $(p_0 - \sqrt{2})^2 > 0$ whenever $p_0 \ne \sqrt{2}$ to show that if $0 < p_0 < \sqrt{2}$, then $p_1 > \sqrt{2}$. (Hint: Show $p_1 - \sqrt{2} = \frac{(p_0 - \sqrt{2})^2}{2p_0}$.)

(e) Using (c) and (d) to show $\lim_{n\to\infty} p_n = \sqrt{2}$ for any $p_0 > 0$.

Solution:

(a) $g(x) = \frac{x}{2} + \frac{1}{x}$.

(b) For any $p_0 > \sqrt{2}$, since $\frac{p_{n-1}}{2} \ne \frac{1}{p_{n-1}}$, we have

$$p_n = \frac{p_{n-1}}{2} + \frac{1}{p_{n-1}} > 2\sqrt{\frac{p_{n-1}}{2}\frac{1}{p_{n-1}}} = \sqrt{2}.$$

Moreover, for any $p_0 > \sqrt{2}$, $\frac{p_{n-1}}{2} > \frac{1}{p_{n-1}}$ and $p_n < \frac{p_{n-1}}{2} + \frac{p_{n-1}}{2} = p_{n-1}$. p_n is a decreasing sequence and has a lower bound. Therefore, we have $\lim_{n\to\infty} p_n = p^*$ exists. Taking a limit on

$$p_n = \frac{p_{n-1}}{2} + \frac{1}{p_{n-1}}, \quad n = 1, 2, \cdots,$$

we have $p^* = \frac{p^*}{2} + \frac{1}{p^*}$ and $p^* = \sqrt{2}$.

(c) For $0 < p_0 < \sqrt{2}$, we have

$$p_1 = \frac{p_0}{2} + \frac{1}{p_0} = \frac{p_0^2 + 2}{2p_0} = \frac{(p_0 - \sqrt{2})^2}{2p_0} + \sqrt{2} > \sqrt{2}.$$

(d) Therefore, we have for any $p_0 > 0$, $\lim_{n\to\infty} p_n = \sqrt{2}$.