Math 781 Hw5 Solution

1. Perform four iterations of Newton's method for the polynomial

$$f(x) = 4x^3 - 2x^2 + 3$$

starting with $p_0 = -1$.

Solution: $f'(x) = 12x^2 - 4x$ and

$$p_n = p_{n-1} - \frac{4p_{n-1}^3 - 2p_{n-1}^2 + 3}{12p_{n-1}^2 - 4p_{n-1}}.$$

$p_0 = 1$, $p_1 = -0.8125$, $p_2 = -0.7708$, $p_3 = -0.7688$, $p_4 = -0.7688$.

2. Devise a Newton's algorithm for computing the fifth root of any positive number.

Solution: We would like to compute $\sqrt[5]{a}$, $a > 0$. Let $f(x) = x^5 - a$ and $f'(x) = 5x^4$. The Newton's algorithm is

$$p_n = p_{n-1} - \frac{p_{n-1}^5 - a}{5p_{n-1}^4}.$$

3. Suppose that p is a double zero of the function f. Thus $f(p) = f'(p) = 0 \neq f''(p)$. Show that if f'' is continuous, then in Newton's method we shall have

$$\lim_{n\to\infty} \frac{|p - p_n|}{|p - p_{n-1}|} = \frac{1}{2}.$$

Proof.

$$f(p) = f(p_{n-1}) + f'(p_{n-1})(p - p_{n-1}) + f''(\xi)\frac{(p - p_{n-1})^2}{2},$$

where ξ is between p and p_{n-1}. We obtain

$$f(p_{n-1}) = -f'(p_{n-1})(p - p_{n-1}) - f''(\xi)\frac{(p - p_{n-1})^2}{2},$$

We have

$$\begin{aligned}
\frac{p - p_n}{p - p_{n-1}} &= \frac{p - p_{n-1} + \frac{f(p_{n-1})}{f'(p_{n-1})}}{p - p_{n-1}} \\
&= \frac{(p - p_{n-1})f'(p_{n-1}) + f(p_{n-1})}{(p - p_{n-1})f'(p_{n-1})} \\
&= \frac{(p - p_{n-1})f'(p_{n-1}) - f'(p_{n-1})(p - p_{n-1}) - f''(\xi)\frac{(p-p_{n-1})^2}{2}}{(p - p_{n-1})f'(p_{n-1})} \\
&= f''(\xi)\frac{p - p_{n-1}}{2f'(p_{n-1})} \to \frac{1}{2}(n \to \infty.)
\end{aligned}$$

4. Suppose $f(x) = (x-p)^k h(x)$, where $k \geq 1$ is an integer, $h(p) \neq 0$ and $h'''(x)$ is continuous in a neighborhood of p. Prove the modified Newton's method

$$p_n = p_{n-1} - \frac{kf(p_{n-1})}{f'(p_{n-1})}$$

converges at least quadratically. (Hint: Use the fixed point iteration result to show $\lim_{x \to p} g'(x) = 0$.)

Proof. $g(x) = x - \frac{kf(x)}{f'(x)}$. We have

$$\begin{aligned}
g'(x) &= 1 - \left(k - \frac{kf(x)f''(x)}{[f'(x)]^2}\right) = 1 - k + k\frac{f(x)f''(x)}{[f'(x)]^2} \\
&= 1 - k + k\frac{(x-p)^k h(x)[k(k-1)(x-p)^{k-2}h(x) + 2k(x-p)^{k-1}h'(x) + (x-p)^k h''(x)]}{[k(x-p)^{k-1}h(x) + (x-p)^k h'(x)]^2} \\
&= 1 - k + k\frac{h(x)[k(k-1)h(x) + 2k(x-p)h'(x) + (x-p)^2 h''(x)]}{[kh(x) + (x-p)h'(x)]^2} \\
&\to 1 - k + k\frac{k(k-1)}{k^2} = 0.(x \to p)
\end{aligned}$$