Math 781 Hw6 Solution

1. If a secant method is applied to the function $f(x) = x^2 - 2$, with $x_0 = 0$ and $x_1 = 1$. What is x_2? *Solution*: $x_0 = 0$, $x_1 = 1$, $f(x_0) = 0 - 2 = -2$, $f(x_1) = 1^2 - 2 = -1$

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)} = 1 - \frac{(-1)(1-0)}{-1-(-2)} = 1 + 1 = 2.$$

2. Find the polynomial of least degree that interpolate the set of data.

x	7	1	2
y	146	2	1

Solution:

$x_0 = 7$, $x_1 = 1$, and $x_2 = 2$.

$$\begin{aligned} L_0 &= \frac{(x-1)(x-2)}{(7-1)(7-2)} = \frac{(x-1)(x-2)}{30}, \\ L_1 &= \frac{(x-7)(x-2)}{(1-7)(1-2)} = \frac{(x-7)(x-2)}{6}, \\ L_2 &= \frac{(x-7)(x-1)}{(2-7)(2-1)} = -\frac{(x-7)(x-1)}{5}, \end{aligned}$$

$$P_2(x) = 146L_0 + 2L_1 + L_2 = \frac{146}{30}(x-1)(x-2) + \frac{1}{3}(x-7)(x-2) - \frac{1}{5}(x-7)(x-1).$$

3. Prove that if g interpolates the function f at $x_0, x_1, \cdots, x_{n-1}$ and if h interpolates f at $x_1, x_2, \cdots, x_n$, then the function

$$g(x) + \frac{x_0 - x}{x_n - x_0}(g(x) - h(x))$$

interpolate f at $x_0, x_1, \cdots, x_n$. (Hint: You need to check the function value at x_i equals to $f(x_i)$.).

Proof. Since g interpolates the function f at $x_0, x_1, \cdots, x_{n-1}$, we have $g(x_i) = f(x_i)$ for $i = 0, \cdots, n-1$. Similarly, we have $h(x_i) = f(x_i)$ for $i = 1, \cdots, n$.

For $i = 0$, we have

$$g(x_0) + \frac{x_0 - x_0}{x_n - x_0}(g(x_0) - h(x_0)) = f(x_0) + 0 = f(x_0);$$

For $i = 1, \cdots, n-1$, we have $g(x_i) = h(x_i) = f(x_i)$ and

$$g(x_i) + \frac{x_0 - x_i}{x_n - x_0}(g(x_i) - h(x_i)) = f(x_i) + 0 = f(x_i);$$

For $i = n$, we have $h(x_n) = f(x_n)$ and

$$g(x_n) + \frac{x_0 - x_n}{x_n - x_0}(g(x_n) - h(x_n)) \quad = \quad g(x_n) - (g(x_n) - f(x_n)) = f(x_n).$$

4. Prove that $\sum_{i=0}^{n} L_i(x) = 1$ for all x, where $L_i(x)$ is the Lagrange interpolation polynomial basis. (Hint: Consider the interpolation polynomial for the function $f(x) = 1$.)

   Proof. Let $f(x) = 1$ and $P_n(x)$ be the interpolation polynomial which interpolates $f(x)$ at $x_0, \cdots x_n$. Using the Lagrange interpolation basis, we have

   $$P_n(x) = \sum_{i=0}^{n} f(x_i) L_x(x).$$

   Since $f(x) = 1$, $P_0 = 1$ interpolates $f(x)$ at $x_0, \cdots x_n$. Due to the uniqueness of the interpolation polynomial, we have $P_n(x) = 1$ and $f(x_i) = 1$, we have

   $$1 = \sum_{i=0}^{n} L_x(x).$$