Math 781 Hw7 Solution

1. Prove that if f is a polynomial degree k, then for $n > k$,

$$f[x_0, x_1, \cdots, x_n] = 0.$$

Proof. Let $P_n(x)$ be the interpolation polynomial interpolating $f(x)$ at $x_0, \cdots, x_n$. $f[x_0, x_1, \cdots, x_n]$ is the coefficient of x^n in $P_n(x)$. Since $f(x)$ is a polynomial of degree $\leq n$ and $f(x_i) = f(x_i)$ for $i = 0, \cdots, n$, $f(x)$ is the interpolation polynomial interpolating $f(x)$ at $x_0, \cdots, x_n$. Due to the uniqueness of the interpolation polynomial, $P_n(x) = f(x)$ and the coefficient of x^n is zero since the degree of $f(x)$ is less than n. We have $f[x_0, x_1, \cdots, x_n] = 0$.

2. Let $L_i(x)$ be the Lagrange interpolation polynomial basis which interpolates a function at $x_0, \cdots x_n$. Show for any f,

$$\sum_{i=0}^{n} f(x_i)L_i(x) = \sum_{i=0}^{n} f[x_0, \cdots, x_i]\Pi_{j=0}^{i-1}(x - x_j).$$

Use this result to show

$$f[x_0, \cdots, x_n] = \sum_{i=0}^{n} f(x_i)\Pi_{j=0, j\neq i}^{n}(x_i - x_j)^{-1}.$$

Proof. Let $P_n(x)$ be the interpolation polynomial interpolating $f(x)$ at $x_0, \cdots, x_n$. Using the Lagrange interpolation polynomial basis, we can write $P_n(x) = \sum_{i=0}^{n} f(x_i)L_i(x)$. On the other hand, using the Newton interpolation polynomial, we have $P_n(x) = \sum_{i=0}^{n} f[x_0, \cdots, x_i]\Pi_{j=0}^{i-1}(x - x_j)$. Due to the uniqueness of the interpolation polynomial, we have

$$\sum_{i=0}^{n} f(x_i)L_i(x) = \sum_{i=0}^{n} f[x_0, \cdots, x_i]\Pi_{j=0}^{i-1}(x - x_j).$$

Since $f[x_0, \cdots, x_n]$ is the coefficient of x^n in $P_n(x)$. $\sum_{i=0}^{n} f(x_i)\Pi_{j=0, j\neq i}^{n}(x_i - x_j)^{-1}$ is also the coefficient of x^n of $P_n(x)$ in the Lagrange form. Therefore

$$f[x_0, \cdots, x_n] = \sum_{i=0}^{n} f(x_i)\Pi_{j=0, j\neq i}^{n}(x_i - x_j)^{-1}.$$

3. Determine the Newton interpolation polynomial for this table:

x	0	1	2	7
y	51	3	1	201

Solution: We have

x	$f[x_i]$			
0	51			
1	3	$f[0,1] = \frac{3-51}{1-0} = -48$		
2	1	$f[1,2] = \frac{1-3}{2-1} = -2$	$f[0,1,2] = \frac{-2+48}{2-0} = 23$	
7	201	$f[2,7] = \frac{201-1}{7-2} = 40$	$f[1,2,7] = \frac{40+2}{7-1} = 7$	$f[0,1,2,7] = \frac{7-23}{7-0} = -\frac{16}{7}$

Therefore $P_3(x) = 51 - 48(x-0) + 23(x-0)(x-1) - \frac{16}{7}(x-0)(x-1)(x-2)$.

4. Obtain a formula for the polynomial p of least degree that takes these values:

$$p(x_i) = y_i, \quad p'(x_i) = 0, \quad , i = 0, \cdots, n.$$

(Hint: use the Hermite interpolation polynomials).

Solution: We have

$$P_{2n+1}(x) = \sum_{i=0}^{n} y_i H_i(x) + \sum_{i=0}^{n} p'(x_i)\hat{H}_i(x) = \sum_{i=0}^{n} y_i H_i(x).$$