Math 781 Hw8 Solution

1. Use the extended Newton divided difference method to obtain a quartic polynomial that takes these values:

x	0	1	2
$p(x)$	2	-4	44
$p'(x)$	-9	4	

Solution: We have

z	$f[z_i]$				
0	2				
0	2	-9			
1	-4	$\frac{-4-2}{1-0} = -6$	$\frac{-6+9}{1-0} = 3$		
1	-4	4	$\frac{4+6}{1-0} = 10$	$\frac{10-3}{1-0} = 7$	
2	44	$\frac{44+4}{2-1} = 48$	$\frac{48-4}{2-1} = 44$	$\frac{44-10}{2-0} = 17$	$\frac{17-7}{2-0} = 5$

Therefore $P_4(x) = 2-9(x-0)+3(x-0)(x-0)+7(x-0)^2(x-1)+5(x-0)^2(x-1)^2 = 2 - 9x + 3x^2 + 7x^2(x-1) + 5x^2(x-1)^2$.

2. Determine whether this is a quadratic spline function:

$$f(x) = \begin{cases} x & x \in (-\infty, 1]; \\ -\frac{1}{2}(2-x)^2 + \frac{3}{2} & x \in [1, 2]; \\ \frac{3}{2} & x \in [2, \infty). \end{cases}$$

Solution: On each interval, $f(x)$ is a polynomial degree ≤ 2.

(a) At $x = 1$:
$f(x) = 1$ on $(-\infty, 1]$ and $f(1) = -\frac{1}{2}(2-1)^2 + \frac{3}{2} = 1$ on $[1, 2]$,
therefore $f(x)$ is continuous at $x = 1$;
$f'(x) = 1$ on $(-\infty, 1]$ and $f'(1) = -\frac{1}{2}2(2-1)(-1) = 1$ on $[1, 2]$,
therefore $f'(x)$ is continuous at $x = 1$;

(b) At $x = 2$:
$f(x) = -\frac{1}{2}(2-2)^2 + \frac{3}{2} = \frac{3}{2}$ on $[1, 2]$ and $f(1) = \frac{3}{2}$ on $[2, \infty)$,
therefore $f(x)$ is continuous at $x = 2$;
$f'(2) = -\frac{1}{2}2(2-2)(-1) = 0$ on $[1, 2]$, and $f'(2) = 0$ on $[2, \infty)$,
therefore $f'(x)$ is continuous at $x = 2$;

This is a quadratic spline function.

3. Determine whether this is a natural cubic spline:

$$f(x) = \begin{cases} 2(x+1) + (x+1)^3 & x \in [-1, 0]; \\ 3 + 5x + 3x^2 & x \in [0, 1]; \\ 11 + 11(x-1) + 3(x-1)^2 - (x-1)^3 & x \in [1, 2]. \end{cases}$$

Solution: On each interval, $f(x)$ is a polynomial degree ≤ 3.

(a) At $x = 0$:
$f(0) = 2(0+1) + (0+1)^3 = 5$ on $[-1, 0]$ and $f(0) = 3 + 5(0) + 3(0)^2 = 3$ on $[0, 1]$,
therefore $f(x)$ is continuous at $x = 0$;
$f'(0) = 2(0) + 3(0+1)^2 = 3$ on $[-1, 0]$ and $f'(0) = 5 + 6(0) = 5$ on $[0, 1]$,
therefore $f'(x)$ is continuous at $x = 0$;
$f''(0) = 6(0+1) = 6$ on $[-1, 0]$ and $f''(0) = 6$ on $[0, 1]$,
therefore $f''(x)$ is continuous at $x = 0$;

(b) At $x = 1$:
$f(1) = 3 + 5(1) + 3(1)^2 = 11$ on $[0, 1]$,
$f(1) = 11 + 11(1-1) + 3(1-1)^2 - (1-1)^3 = 11$ on $[1, 2]$,
therefore $f(x)$ is continuous at $x = 1$;
$f'(1) = 5 + 6(1) = 11$ on $[0, 1]$,
$f'(1) = 11 + 6(1-1) - 3(1-1)^2 = 11$ on $[1, 2]$,
therefore $f'(x)$ is continuous at $x = 1$;
$f''(1) = 6$ on $[0, 1]$,
$f''(1) = 6 - 6(1-1) = 6$ on $[1, 2]$,
therefore $f''(x)$ is continuous at $x = 1$;

(c) $f''(-1) = 6(-1+1) = 0$ and $f''(2) = 6(2-1) - 6(2-1) = 0$

This is a natural cubic spline function.

4. Determine the values of (a, b, c) that makes the function

$$f(x) = \begin{cases} x^3 & x \in [0, 1]; \\ \frac{1}{2}(x-1)^3 + a(x-1)^2 + b(x-1) + c & x \in [1, 3] \end{cases}$$

a cubic spline. Is it a natural cubic spline?

Solution:

(a) At $x = 1$:
$f(1) = 1^3 = 1$ on $[0, 1]$,
$f(1) = c$ on $[1, 3]$,
since $f(x)$ is continuous at $x = 1$, therefore $c = 1$.
$f'(1) = 3$ on $[0, 1]$,
$f'(1) = b$ on $[1, 3]$,
since $f'(x)$ is continuous at $x = 1$, therefore $b = 3$.
$f''(1) = 6$ on $[0, 1]$,
$f'(1) = 2a$ on $[1, 3]$,
since $f''(x)$ is continuous at $x = 1$, therefore $2a = 6$ and $a = 3$.

(b) $f''(0) = 6(0) = 0$ and $f''(3) = 3(3 - 1) + 2a = 12 \neq 0$
Therefore it is not a natural cubic spline.