# Math 781 Hw9 Solution

1. Determine the degree 5 Chebyshev polynomial $T_5(x)$ using the three-term recurrence.

   *Solution*: $T_0(x) = 1$, $T_1(x) = x$,

$$\begin{aligned} T_2(x) &= 2xT_1(x) - T_0(x) = 2x^2 - 1 \\ T_3(x) &= 2xT_2(x) - T_1(x) = 2^2x^3 - 3x \\ T_4(x) &= 2xT_3(x) - T_2(x) = 2^3x^4 - 6x^2 - 2x^2 + 1 = 2^3x^4 - 8x^2 + 1 \\ T_5(x) &= 2xT_4(x) - T_3(x) = 2^4x^5 - 16x^3 + 2x - 2^2x^3 + 3x = 2^4x^5 - 20x^3 + 5x. \end{aligned}$$

2. Find the linear least squares approximation to $f(x) = \sin(x)$ on the interval $[-1, 1]$.

   *Solution*: $f(x) = \sin x$. $P_1(x) = \alpha_0 Q_0(x) + \alpha_1 Q_1(x)$, where $Q_0 = 1$, $Q_1 = x$ are first two Legendre polynomials. We have

$$\begin{aligned} \alpha_i &= \frac{2i+1}{2}\int_{-1}^{1} f(x)Q_i(x)dx, \quad i = 0, 1 \\ \alpha_0 &= \frac{2\cdot 0+1}{2}\int_{-1}^{1} \sin x dx = \frac{1}{2}(-\cos x)|_{-1}^{1} = 0 \\ \alpha_1 &= \frac{2\cdot 1+1}{2}\int_{-1}^{1} \sin x \cdot x dx = 3(\sin 1 - \cos 1). \end{aligned}$$

   Therefore $P_1(x) = 0 \cdot Q_0(x) + 3(\sin 1 - \cos 1)Q_1(x) = 0.9035x$.

3. Let $w(x) = \frac{1}{\sqrt{1-x^2}}$ for $-1 < x < 1$. Show that the Chebyshev polynomials $T_n(x) = \cos(n \arccos x)$ $(n \geq 0)$ satisfy

$$\int_{-1}^{1} w(x)T_i(x)T_j(x)\ dx = 0, \quad \forall i \neq j,$$

   and dtermine $\int_{-1}^{1} w(x)\left(T_n(x)\right)^2\ dx$.

   Hint: Use substitution with $x = \cos\theta$ with $\theta \in [0, \pi]$.

   *Proof.* Let $x = \cos\theta$, for $\theta \in [0, \pi]$. $dx = -\sin\theta d\theta$ and $\theta = arccosx$. Therefore, we have

$$\begin{aligned} \int_{-1}^{1} w(x)T_i(x)T_j(x)\ dx &= \int_0^{\pi} \cos(i\theta)\cos(j\theta)d\theta \\ &= \int_0^{\pi} \frac{1}{2}\cos((i-j)\theta) + \frac{1}{2}\cos((i+j)\theta)d\theta \\ &= \frac{1}{2}\int_0^{\pi} \frac{1}{2}\cos((i-j)\theta)d\theta + \frac{1}{2}\int_0^{\pi} \frac{1}{2}\cos((i+j)\theta)d\theta \\ &= \frac{1}{2}\int_0^{\pi} \frac{1}{2}\cos((i-j)\theta)d\theta + \frac{1}{2}\frac{\sin((i+j)\theta)}{i+j}|_0^{\pi} \\ &= \frac{1}{2}\int_0^{\pi} \frac{1}{2}\cos((i-j)\theta). \end{aligned}$$

When $i = j$,

$$\frac{1}{2}\int_0^{\pi}\frac{1}{2}\cos((i-j)\theta)d\theta = \frac{1}{2}\int_0^{\pi} 1 d\theta = \frac{\pi}{2};$$

when $i \neq j$,

$$\frac{1}{2}\int_0^{\pi}\frac{1}{2}\cos((i-j)\theta) = \frac{\sin((i-j)\theta)}{i-j}|_0^{\pi} = 0.$$

Therefore,

$$\int_{-1}^{1} w(x)\left(T_n(x)\right)^2 \ dx = \frac{\pi}{2}.$$