Math 782 Hw2

due Tuesday 01/30/2018

1. Suppose the set of vectors $\{v_1, \ldots, v_n\} \subset \mathbb{C}^m$ is orthogonal. Prove the Pythagorean Theorem:
$$\|v_1 + \cdots v_n\|^2 = \|v_1\|^2 + \cdots \|v_n\|^2.$$

2. Prove that if a matrix A is both upper triangular and unitary, then it is diagonal and $|a_{ii}| = 1$ for all i.

3. A matrix $S \in \mathbb{C}^{m\times m}$ is skew-Hermitian if $S = -S^*$. Suppose S is skew-Hermitian.

   (a) Prove that $I_m - S$ is nonsingular.

   (b) Prove that the matrix $Q = (I_m - S)^{-1}(I_m + S)$ is unitary.

4. Prove that for any $x \in \mathbb{C}^m$ and $A \in \mathbb{C}^{m\times n}$,

   (a) $\|x\|_\infty \le \|x\|_2 \le \sqrt{m}\|x\|_\infty$,

   (b) $\frac{1}{\sqrt{m}}\|x\|_1 \le \|x\|_2 \le \|x\|_1$,

   (c) $\frac{1}{\sqrt{n}}\|A\|_\infty \le \|A\|_2 \le \sqrt{m}\|A\|_\infty$,

   (d) $\frac{1}{\sqrt{m}}\|A\|_1 \le \|A\|_2 \le \sqrt{n}\|A\|_1$.
   Hint: Try to use Properties (3), (4) or (5) for the first inequality in (c) and the second inequality in (d).

5. Let $A = uv^* \in \mathbb{C}^{m\times n}$ with $u \in \mathbb{C}^m$ and $v \in \mathbb{C}^n$. Prove

   (a) $\|A\|_1 = \|u\|_1\|v\|_\infty$,

   (b) $\|A\|_\infty = \|u\|_\infty\|v\|_1$.