

# HW2

(1)

1. Proof.  $\|v_1 + v_2 + \dots + v_n\|^2$   
 $= (v_1 + v_2 + \dots + v_n)^* (v_1 + v_2 + \dots + v_n)$

$$\underbrace{v_i^* v_j = 0}_{i \neq j} \quad v_1^* v_1 + v_2^* v_2 + \dots + v_n^* v_n = \|v_1\|^2 + \|v_2\|^2 + \dots + \|v_n\|^2$$

2. Since  $A$  is unitary matrix.  $A$  is square.  $A$  is upper triangle

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ 0 & \cdots & & a_{nn} \end{pmatrix}_{n \times n}$$

$$A^* A = I_n \Rightarrow \begin{pmatrix} \bar{a}_{11} & 0 & \cdots & 0 \\ \bar{a}_{12} & \bar{a}_{22} & \cdots & 0 \\ \vdots & & \ddots & \\ \bar{a}_{1n} & \bar{a}_{2n} & \cdots & \bar{a}_{nn} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ 0 & \cdots & & a_{nn} \end{pmatrix} = I_n$$

$$\bar{a}_{11} \cdot a_{11} = |a_{11}|^2 = 1 \quad \bar{a}_{11} \neq 0 \quad a_{12} \cdots a_{1n} = 0 \quad \text{to match the first row of } I_n$$

$$\text{similarly to match the second row of } I_n. \quad \bar{a}_{22} \cdot a_{22} = |a_{22}|^2 = 1 \\ a_{23} \cdots a_{2n} = 0$$

$$\bar{a}_{nn} \cdot a_{nn} = |a_{nn}|^2 = 1$$

Therefore  $A$  is diagonal and  $|a_{ii}| = 1$  for all  $i$

(2)

3. Proof:

(a) let  $x \in \mathbb{C}^m$  such that  $(I_m - S)x = 0$ 

$$(I_m - S)x = 0 \Rightarrow Sx = x \Rightarrow x^* S^* = x^* \Rightarrow x^* S^* x = x^* x$$

$$\stackrel{S^* = -S}{\Rightarrow} -x^* S x = x^* x \stackrel{Sx=x}{\Rightarrow} -x^* x = x^* x \text{ i.e. } -\|x\|^2 = \|x\|^2 \Rightarrow \|x\|^2 = 0 \Rightarrow x = 0$$

therefore  $I_m - S$  is nonsingular(b)  $Q$  is a square matrix

$$\begin{aligned} Q Q^* &= (I_m - S)^{-1} (I_m + S) (I_m + S)^* ((I_m - S)^{-1})^* \\ &= (I_m - S)^{-1} [(I_m + S)(I_m - S)] ((I_m - S)^{-1})^* \\ &= (I_m - S)^{-1} [I_m - S \cdot S] ((I_m - S)^{-1})^* \\ &= (I_m - S)^{-1} (I_m - S)(I_m + S)((I_m - S)^{-1})^* \\ &= (I_m + S)((I_m - S)^{-1})^* \\ &= \left[ (I_m - S)^{-1} (I_m + S)^* \right]^* = \left[ (I_m - S)^{-1} (I_m - S) \right]^* = I \end{aligned}$$

therefore  $Q$  is unitary

(3)

4. Proof: Let  $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$   $x_k = \max_{1 \leq i \leq m} |x_i|$

$$(a) \|x\|_\infty^2 = \left( \max_{1 \leq i \leq m} |x_i| \right)^2 = |x_k|^2 \leq \sum_{i=1}^m |x_i|^2 = \|x\|_2^2 \Rightarrow \|x\|_\infty \leq \|x\|_2$$

$$\|x\|_2^2 = \sum_{i=1}^m |x_i|^2 \leq \sum_{i=1}^m |x_k|^2 = m |x_k|^2 = m \|x\|_\infty^2 \Rightarrow \|x\|_2 \leq \sqrt{m} \|x\|_\infty$$

$$(b) \|x\|_1^2 = \left( \sum_{i=1}^m |x_i| \right)^2 = \sum_{i,j=1}^m |x_i x_j| \leq \sum_{i,j=1}^m \frac{|x_i|^2 + |x_j|^2}{2} = m \sum_{i=1}^m |x_i|^2 = m \|x\|_2^2$$

$$\Rightarrow \|x\|_1 \leq \sqrt{m} \|x\|_2$$

$$\|x\|_2^2 = \sum_{i=1}^m |x_i|^2 \leq \sum_{i,j=1}^m |x_i x_j| = \|x\|_1^2 \Rightarrow \|x\|_2 \leq \|x\|_1$$

$$(c, d) \|A\|_2 = \max_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \frac{\|Ax\|_2}{\|x\|_2} \stackrel{(a)}{\leq} \max_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \frac{\sqrt{m} \|Ax\|_\infty}{\|x\|_2} \stackrel{(a)}{\leq} \max_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \frac{\sqrt{m} \|Ax\|_\infty}{\|x\|_\infty} = \sqrt{m} \|A\|_\infty$$

$$\|A\|_2 = \max_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \frac{\|Ax\|_2}{\|x\|_2} \stackrel{(b)}{\geq} \max_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \frac{\frac{1}{\sqrt{m}} \|Ax\|_1}{\|x\|_2} \stackrel{(b)}{\geq} \max_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \frac{1}{\sqrt{m}} \frac{\|Ax\|_1}{\|x\|_1} = \frac{1}{\sqrt{m}} \|A\|_1$$

$$\|A\|_2 = \|A^*\|_2 \geq \frac{1}{\sqrt{n}} \|A^*\|_1 = \frac{1}{\sqrt{n}} \|A\|_1$$

$$\|A\|_2 = \|A^*\|_2 \leq \sqrt{n} \|A^*\|_\infty = \sqrt{n} \|A\|_\infty$$

Therefore  $\frac{1}{\sqrt{n}} \|A\|_\infty \leq \|A\|_2 \leq \sqrt{m} \|A\|_\infty$

$$\frac{1}{\sqrt{m}} \|A\|_1 \leq \|A\|_2 \leq \sqrt{n} \|A\|_\infty$$

(4)

5. Proof:

$$(a) \|A\|_1 = \max_{\substack{x \in \mathbb{C}^n \\ \|x\|_1=1}} \|Ax\|_1 = \max_{\substack{x \in \mathbb{C}^n \\ \|x\|_1=1}} \|Uv^*x\|_1 = \max_{\substack{x \in \mathbb{C}^n \\ \|x\|_1=1}} |v^*x| \|U\|_1$$

$v^*x$  is a scalar

for any  $\|x\|_1=1$   $|v^*x| = \left| \sum_{i=1}^n v_i x_i \right| \leq \sum_{i=1}^n |v_i| |x_i| \leq \|V\|_\infty \sum_{i=1}^n |x_i| = \|V\|_\infty$

therefore  $\|A\|_1 \leq \|U\|_1 \|V\|_\infty$

on the other hand let  $|V_k| = \|V\|_\infty$  take  $x = e_k$   $\|x\|_1=1$

$$|v^*x| = |v_k| = \|V\|_\infty$$

$$\|A\|_1 = \max_{\substack{x \in \mathbb{C}^n \\ \|x\|_1=1}} |v^*x| \|U\|_1 \geq |v^*e_k| \|U\|_1 = |v_k| \|U\|_1 = \|V\|_\infty \|U\|_1$$

therefore  $\|A\|_1 = \|U\|_1 \cdot \|V\|_\infty$

$$(b) \|A\|_\infty = \|A^*\|_1 = \|V^*U\|_1 = \|V\|_\infty \cdot \|U\|_\infty$$