Math 782 Hw5

due Tuesday 02/20/2018

1. Suppose $\{q_1, \cdots, q_n\} \subset \mathbb{R}^m$ is orthonormal. Let $Q = [q_1 \cdots q_n]$. Prove
$$I_m - QQ^* = (I_m - q_n q_n^*)(I_m - q_{n-1}q_{n-1}^*) \cdots (I_m - q_1 q_1^*).$$

2. Suppose $0 \neq v \in \mathbb{C}^k$ and a Householder reflector $H(v)$ is defined as $H(v) = I_k - 2\frac{vv^*}{v^*v}$. Let
$$Q = \begin{bmatrix} I_{m-k} & 0 \\ 0 & H(v) \end{bmatrix} \in \mathbb{C}^{m\times m}.$$
Prove
$$Q = I_m - 2\frac{\hat{v}\hat{v}^*}{\hat{v}^*\hat{v}}, \quad \hat{v} = \begin{bmatrix} 0_{(m-k)\times 1} \\ v \end{bmatrix},$$
i.e., Q is an $m \times m$ Householder reflector.

3. Prove that H is an $m\times m$ Householder reflector if and only if there is a unitary matrix Q such that
$$H = Q\begin{bmatrix} I_{m-1} & \\ & -1 \end{bmatrix} Q^*.$$
Hint: For "only if" part, you may use $H = H^*$ to get a spectral decomposition of H and use $H = H^{-1}$ to show the eigenvalues can only be ± 1. Finally you can use $I - H$ is a rank-1 matrix to show that -1 is a single eigenvalue of H.

   For "if" part, show $H = I_m - 2q_m q_m^*$, where q_m is the last column of Q.

4. Suppose H is a Householder reflector. Determine the determinant of H and the singular values of H.

   Hint: Use the result in the previous problem.