Math 782 Hw6

due Tuesday 03/06/2018

1. Let
$$x = \begin{bmatrix} 1 \\ 4 \\ 7 \\ 4 \\ 4 \end{bmatrix}, \qquad e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

   (a) Determine a Householder matrix $H = I - vv^T/\beta$ that transforms x to a scalar multiple of e_1. You only need to provide the vector v and the scalar β.

   (b) Compute the product Hx with your H.

2. Suppose $A \in \mathbb{F}^{m\times n}$ has the full column rank ($\text{rank}(A) = n$). Prove A^*A has a Cholesky factorization. That is, there exists an invertible upper triangular matrix $R \in \mathbb{F}^{n\times n}$ such that $A^*A = R^*R$.

   Hint: Use a reduced QR factorization of A.

3. Suppose $A \in \mathbb{F}^{m\times n}$ has the full column rank ($\text{rank}(A) = n$) and $b \in \mathbb{F}^m$. Let
$$A = QR, \quad Q = \begin{bmatrix} \widehat{Q} & \widetilde{Q} \end{bmatrix}, \quad R = \begin{bmatrix} \widehat{R} \\ 0 \end{bmatrix}$$
   be a full QR factorization of A, where Q is unitary, $\widehat{Q} \in \mathbb{F}^{m\times n}$, and $\widehat{R} \in \mathbb{F}^{n\times n}$ is upper triangular. Define
$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} := Q^*b = \begin{bmatrix} \widehat{Q}^*b \\ \widetilde{Q}^*b \end{bmatrix}.$$
   If $x \in \mathbb{F}^n$ is the solution to the least squares problem $\min_{x\in\mathbb{F}^n} \|b - Ax\|_2$ and $r = b - Ax$, Prove

   (a) $Ax = \widehat{Q}b_1$ and $\widehat{R}x = b_1$

   (b) $r = \widetilde{Q}b_2$ and $\|r\|_2 = \|b_2\|_2$.