# Math 782 Hw9

due Tuesday 04/10/2018

1. (Optional, you do not need to do this) Let$A \in \mathbb{C}^{m\times m}$ be nonsingular. Prove that $A$ has an LU factorization if and only if for every $k$ with $1 \le k \le m$, the leading principal $k \times k$ submatrix $A_k$ , as shown in the partitioning
$$A = \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix},$$
is nonsingular.

   *Hint: For the necessity, by induction, assume after $k$ steps we have*
$$L_k \cdots L_1 A = \begin{bmatrix} U_k & * & * \\ 0 & a^{(k)}_{k+1,k+1} & * \\ 0 & * & * \end{bmatrix},$$
*where $U_k$ is upper triangular and invertible. By comparing the $(k+1)\times(k+1)$ leading principal submatrices on both sides to show $a^{(k)}_{k+1,k+1} \neq 0$ and the Gaussian elimination process then can continue.*

   *For the sufficiency, for each $1 \le k \le m$, partition $L = \begin{bmatrix} L_k & 0 \\ & * \end{bmatrix}$ and $U = \begin{bmatrix} U_k & * \\ 0 & * \end{bmatrix}$. Show $U_k$ and $L_k$ are invertible and $A_k = L_k U_k$.*

2. Let$A \in \mathbb{C}^{m\times m}$ be nonsingular. Suppose $A$ satisfies the condition given in previous problem, so it has an LU factorization $A = LU$. Prove the factorization is unique. *Hint: Suppose $A$ has two LU factorization $A = L_1U_1 = L_2U_2$. Show that $L_2$ and $U_1$ are invertible and $L_2^{-1}L_1 = U_2U_1^{-1}$. Then try to prove $L_1 = L_2$ and $U_1 = U_2$ by using the triangular structures.*

3. Let
$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 4 & -1 & 6 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ 0 \\ 11 \end{bmatrix}.$$
Obtain the LU factorization of $A$ using the elementary lower triangular matrices and solve the system $Ax = b$.

4. Let $A$ be a symmetric matrix with $a_{11} \neq 0$. Suppose $A$ has been reduced to the form
$$\left[\begin{array}{c|ccc} a_{11} & a_{12} & \cdots & a_{1n} \\ \hline 0 & & & \\ \vdots & & A^{(1)} & \\ 0 & & & \end{array}\right] = L_1 A.$$

Prove that $A^{(1)}$ is symmetric.

*Hint: Denote* $a_1 = \begin{bmatrix} a_{12} \\ \vdots \\ a_{1n} \end{bmatrix}$ *and write* $L_1$ *in term of* $a_1$. *Try to write the expression of* $A^{(1)}$.