Combinatorics for Dad

CJ Harries

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Introduction

Mathematics is a huge subject. By this point, most mathematicians are highly specialized, each working with a very specific set of tools. Each field has its own unique set, with some common themes like logic and brute force across all fields. For the average person, with maybe some algebra or calculus skills, it's a daunting task to approach "new math" or advanced topics. The style is different and it's rare to find a problem that, like most work in algebra and calculus, has a very specific numerical answer that can be easily achieved.

Combinatorics is different. Unlike many other disciplines, which require a solid grasp of the underlying axioms and theorems along with decent computation skill, combinatorics requires only the ability to count and recognize patterns. Like other math fields, it certainly has its own complicated subfields that require years of study, but basic counting problems are all around.

Enumerative combinatorics, or counting things, is the easiest place to begin the study of discrete mathematics. With some basic set theory and a few multiplication tricks, many problems become immediately accessible.

The goal of this text is to introduce the reader to some combinatorial principles and arguments, along with a small amount of graph theory. The text is structured to be similar to lecture notes used in a classroom setting. Unlike reading most fiction or non-fiction, math text requires analyzation, notes, and practice. Each section contains definitions and theorems. Before moving on, it's important to understand what a term means or how a theorem influences the rest of the material. Examples are provided, and it's strongly recommended to attempt as many as necessary to comprehend the material before moving on to a new topic. A large portion of math learning occurs in the process of actively doing it, rather than passively reading or observing it.

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Section 1

Logic

Advanced mathematics uses analytic logic as its driving method. Rather than simply saying, "X is true," mathematicians prove the statement, "X is true," using simple tools.

> **Definition 1.** A **<u>definition</u>** is an if and only if statement statement linking a term to a list of properties.

Definitions are the simplest math tools. Defining a term or object gives you access to certain properties used in proofs and calculations.

A prime number is a number that is divisible only by itself and one. After getting this definition, whenever the word "prime" appears in the text, the list of properties, divisibly only by itself and one, are used. It may seem like over-emphasis of a basic word, but the more complicated a definition gets, the more important it is to mentally plug in the list of properties every time the word appears.

> **Definition 2.** **<u>Axioms</u>** are assumed, underlying mathematical truths, used to build more complicated mathematical objects.

All math begins with a framework of axioms. For example, an axiom you already know is that adding one to a number increases the original by one, e.g. $1+1=2$. Axioms rarely come up in most mathematics, as each is generally assumed to be common knowledge. If a mathematician chose to work in a system where $1+1=0$, he/she would first state that property of the system, so that anyone later reading the work would know the assumed axioms have changed.

Definition 3. **Theorems** are statements about mathematics that are proved using a logical chain of statements, starting with axioms and later involving other theorems or statements. **Corollaries** are immediate results of theorems, and often use either the same proof as the original theorem, or simply use the theorem itself to prove a result. **Lemmas** are small, short theorems usually used to prove broader theorems. **Propositions** are claims that authors have proved, but decided against attaching the weight of the word "theorem" to.

Theorems are the essential tools of mathematics. Each first lists a set of assumptions that must be true for the theorem to apply, then provides a list of results. The proof of a theorem explicitly shows how, starting from the original assumptions, the results are true.

Proposition. 1234567890 is not a prime number.

Proof. 1234567890 is divisible by 2, so, by definition, it cannot be prime □

While simple, this example illustrates the important pieces. Given an assumption, here the definition of prime, something is proved. The proposition itself only contains the important result, the given number is not prime, while the proof goes through the steps to show the result, the number is divisible by 2.

There are more tools necessary to build arguments, but each is introduced later, with examples to ease the reader into the functions of each.

Section 2

Sets

Definition 4. A <u>**set**</u> X is a collection of objects x, written as $X = \{x \in X\}$, where $x \in X$ is read as, "x is in X," or, "x is an element of X."

Typically, sets are denoted with capital letters, like X, while their elements are denoted with lowercase letters, like x. If a set appears, it will be defined so the reader knows exactly what it means. Most mathematical sets have concrete objects, although it is possible to extend the concept to abstract ideas.

Examples:

- $A = \{1, 2, 3\}$: This set contains the elements 1, 2, and 3. $4 \notin A$.
- $X = \{x \text{ is a car}\}$: This set contains any object x such that x is a car. If x is an apple, $x \notin X$. This is an ambiguous set, however, because it is possible to find x such that, by one definition of "car," $x \in X$, but by another, $x \notin X$. (Think about x is a truck. Is this a car?)
- $B = \{a \in A | a \neq 3\}$: This set uses the first example to define its elements, $a \in A$. The "|" symbol, or pipe, is read as "such that." Some mathematicians will replace the pipe with a colon. Either way, it creates a restriction on the elements that can come from the set A: $a \in A$ such that $a \neq 3$. What is an explicit way of writing B? (Hint: write all of the elements of A that are not 3.)

- $\emptyset$: This is a very important set. The "$\emptyset$" denotes the **empty set**, the set containing absolutely no elements. It's used in set comparisons, among other things.

For the purpose of this text, it is important to assume that no element in a set is repeated. For example, $X = \{1, 1\}$ is the same as $X = \{1\}$. Similarly, if $X = \{\text{blue, blue, yellow}\}$, we would instead write $X = \{\text{blue, yellow}\}$. Sets that allow repetition are called **multisets**, but for the most part, this text will avoid multisets.

Definition 5. Given sets X and Y, the **union** $X \cup Y$ is the set containing all the elements of X and all the elements of Y.

Let $X = \{1, 2, 3\}$, $Y = \{4, 5, 6\}$, and $Z = \{7, 8, 9\}$.

$$X \cup Y = \{1, 2, 3, 4, 5, 6\}$$
$$X \cup Y \cup Z = \{x | 1 \leq x \leq 9\}$$

Definition 6. Given sets X and Y, the **intersection** $X \cap Y$ is the set containing all the elements common to both X and Y.

Let $X = \{1, 2, 3\}$, $Y = \{1, 3, 5\}$, and $Z = \{2, 4, 6\}$. $X \cap Y = \{1\}$. $X \cap Z = \{2\}$. $Y \cap Z = \emptyset$. What is $X \cap Y \cap Z$?

Definition 7. Given a set X, a **subset** X' is a set containing some or all of the elements of X, written as $X' \subseteq X$.

Let $X = \{1, 2, 3\}$. Then, if $X' = \{1\}$, $X' \subseteq X$. If Y is a another set, how do X and $X \cap Y$ compare? How do X and $X \cup Y$ compare? (Hint: in both, one is a subset of the other.)

Definition 8. Given a set X, X's **cardinality**, written $|X|$, is the number of elements in X.

Let $X = \{1, 2, 3\}$. $|X| = 3$.

Definition 9. Given a set X, for $X' \subseteq X$, the **compliment** of X' (in X), $X \setminus X'$ or X'^C, is all of the elements of X except for the elements in X'.

Note the new symbol "\". This is read as, "X minus X'," and specifically means the set of of elements in X that are not in X'. If $A, B \subseteq X$, then, while $X \setminus A$ is the compliment of A, $A \setminus B$ is not necessarily the same set. For example, let $X = \{x | 1 \leq x \leq 10\}$, $A = \{1, 3, 5, 7, 9\}$, $B = \{2, 4, 6, 8, 10\}$, and $C = \{1, 3, 6, 9\}$. Here, $X \setminus A = B$, so B is the compliment of A (and A is the compliment of B). However, $X \setminus C = \{2, 4, 5, 7, 8, 10\}$, while $A \setminus C = \{5, 7\}$. What is the compliment of $A \cup C$? $A \cap C$?

As the notation can be confusing, this text will always use A^C to denote the compliment of A.

Definition 10. Given a set X, the **power set** of X, written $\mathcal{P}(X)$, is the set containing all subsets of X. $\mathcal{P}(X) = \{X' | X' \subseteq X\}$.

Let $X = \{1, 2, 3\}$.

$$\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Notice that, unlike the examples before, $\mathcal{P}(X)$ is a set of sets, hence its name. Remember, sets can be collections of anything, including sets.

Definition 11. An ***n*-tuple** is a list of n items $(a_1, a_2, \ldots a_n)$.

n-tuples are used to provide lists of items from multiple sets. For example, if A is a set of colors, B a set of numbers, and C the suits in a deck of cards, the 3-tuple $(\text{red}, 5, \heartsuit)$ could describe the the 5 of hearts.

Definition 12. Given two sets A and B, the **Cartesian product** $A \times B$ is the set of ordered pairs (2-tuples) with the first element from A and the second from B.

$$A \times B = \{(a, b) | a \in A, b \in B\}$$

Cartesian products allow mathematicians to combine multiple sets into a larger set. Unlike unions, the Cartesian product still retains which set each element came from. For example, if $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$,

$$A \times B = \{(1,4),(1,5),(1,6),(2,4),(2,5),(2,6),(3,4),(3,5),(3,6)\}.$$

Cartesian products can be extended to multiple sets. If $A_1, A_2, \ldots, A_n$ are n different sets, then

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } 1 \leq i \leq n\}.$$

2.1 Exercises

1. For each set X below, find the cardinality of X, $|X|$, its powerset $\mathcal{P}(X)$, and find the cardinality of $\mathcal{P}(X)$, $|\mathcal{P}(X)|$. Do you see a pattern?

    a) $X = \{1\}$
    b) $X = \{5, 7\}$
    c) $X = \{\pi, e, i\}$
    d) $X = \{10, 100, 1000, 10000\}$

2. Find the Cartesian product of each of the sets.

    a) $A = \{a\}$, $B = \{1, 2, 3\}$
    b) $A = \{\text{blue, green}\}$, $B = \{5, 6\}$
    c) $A = \{\pi, e, i\}$, $B = \{a, b, c\}$, $C = \{\heartsuit, \spadesuit\}$

Section 3

Important Sets

There are several important sets that mathematicians use regularly.

- $\mathbb{N} = \{1, 2, 3, \ldots\}$. $\mathbb{N}$ is the set of natural numbers. Unlike other sets so far, notice that $\mathbb{N}$ does not list all its elements. Rather, the set definition establishes a pattern, then trails off. Is $47 \in \mathbb{N}$? Is $5.2 \in \mathbb{N}$? Is $-7 \in \mathbb{N}$? (Hint: would the number eventually appear in the counting pattern $1, 2, 3, \ldots$?)

- $\mathbb{Z} = \{0. \pm 1, \pm 2, \pm 3, \ldots\}$. $\mathbb{Z}$ is the set of integers, both positive and negative, including 0. Is $-15 \in \mathbb{Z}$? Is $\pi \in \mathbb{Z}$?

- $\mathbb{Q} = \{\frac{p}{q} | p, q \in \mathbb{Z}, q \neq 0\}$. $\mathbb{Q}$ is the set of rational numbers. It contains any ratio of two integers, with numerator p and denominator q such that $q \neq 0$. Is $\frac{1}{2} \in \mathbb{Q}$? Is $-9 \in \mathbb{Q}$? Is $5.1 \in \mathbb{Q}$? Is $\pi \in \mathbb{Q}$? (Hint: if a number can be expressed as fraction, it's in $\mathbb{Q}$. Remember that some decimals can be written as fractions.)

- $\mathbb{R}$ is the set of real numbers. A precise definition of real numbers is beyond the scope of this text. Think back to algebra. Essentially, anything that is a solution of polynomial with coefficients in $\mathbb{Q}$ is a real number, although many mathematicians would have problems with that definition. For example, $1 \in \mathbb{R}$, $3.01 \in \mathbb{R}$, $-\pi \in \mathbb{R}$, and $\sqrt{2} \in \mathbb{R}$. These are some of the solutions of the equations $x^2 - 2x + 1$, $x^2 - 4.01x + 3.01$, $x^2 - \pi^2$, and $x^2 - 2$.

Given the definitions, notice that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$. This relationship is important, but not so much for combinatorics.

3.1 Exercises

How would you write each of the following sets?

1. All even integers
2. All odd integers
3. Integers over powers of 2
4. Negative real numbers

Section 4

Functions

Definition 13. A **function** f is a map from a set A, called the **domain** of f, to a set B, called the **range** or codomain of f. $f : A \to B$ denotes a function f from A to B.

A function picks an element from A, its domain, and assigns to an element of B, its range. There are many ways to write a function. If $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$, f could be written as

- $f : A \to B$ such that $1 \to 2$, $2 \to 4$, and $3 \to 6$. This explicitly lists the relationship between each value of the domain and range.

- $f : A \to B$ with $f(1) = 2$, $f(2) = 4$, and $f(3) = 6$. Again, this explicitly lists the relationships.

- $f : A \to B$ with $f(a) = 2a$ for $a \in A$. Since f can be written as a formula, this is the simplest and most convenient method to write f. It is important to note that, while one might think $f(5) = 10$, as $5 \notin A$, $f(5)$ is not defined. When using a closed formula for functions, anything not in the domain cannot be run through the function.

Definition 14. An **injective** function $f : A \to B$ is a function that maps every element of its domain to a unique element of its range. In other words, for $a, a' \in A$, if $f(a) = f(a')$, then $a = a'$.

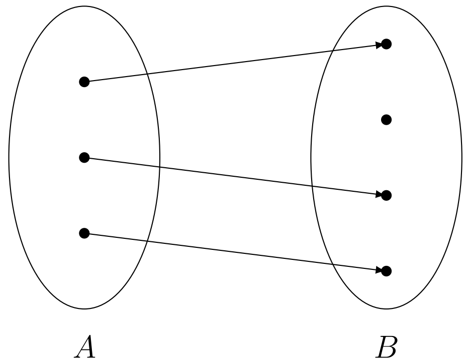

Figure 4.1: An injective function $f : A \to B$

Notice from Figure 4.1 that $|B| > |A|$. The injective condition restricts the size of the domain. There can be infinitely many more elements in the range than in the domain, so long as each element of the domain is sent to a unique element in the range.

Consider $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x$. To prove f is injective, first consider $x, x' \in \mathbb{R}$, and suppose $f(x) = f(x')$, or $x = x'$. While the conclusion was immediate here, the process to show any function is injective is exactly same. Consider two arbitrary elements in the domain, suppose their corresponding elements in the range are equal, and draw a conclusion.

Similarly, consider $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$. Consider again $x, x' \in R$, and suppose $f(x) = f(x')$, or $x^2 = (x')^2$. If $x = 1$ and $x' = -1$, then $1^2 = (-1)^2$, but $1 \neq -1$, so f is not injective.

Definition 15. A **surjective** function $f : A \to B$ is a function such that every element in the range has a corresponding element in the domain, or, for all $b \in B$, $f(a) = b$ for some $a \in A$.

Notice from Figure 4.2 that $|A| > |B|$ and that two elements of the domain are mapped to the same element of the range. This is perfectly acceptable. Surjection is often easier to prove or disprove, especially given an explicit formula for the function. If $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$, then for $b \in \mathbb{R}$, $f(x) = b$ if $x = b$. Ergo f is surjective If $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$, then since there is no element $x \in \mathbb{R}$ such that $x^2 = -1$, f is not surjective.

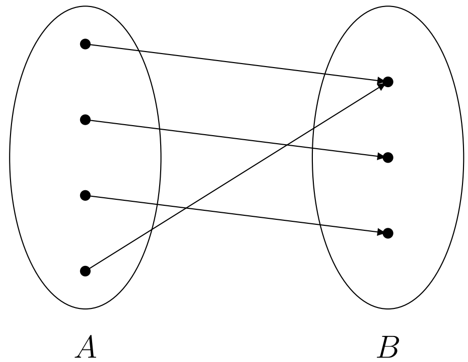

Figure 4.2: A surjective function $f : A \to B$

The last example leads to a common, assumed refinement of range and surjection. A function's **image** is usually defined to be the elements in the range so that f is surjective onto its image. This simplifies the process, and makes most functions, especially those with explicit formulas, surjective. It is still important to check whether or not a function is surjective. It may seem readily apparent more often than not, but sometimes math is easy.

Definition 16. A **bijective** function $f : A \to B$ is a function that is both injective and surjective.

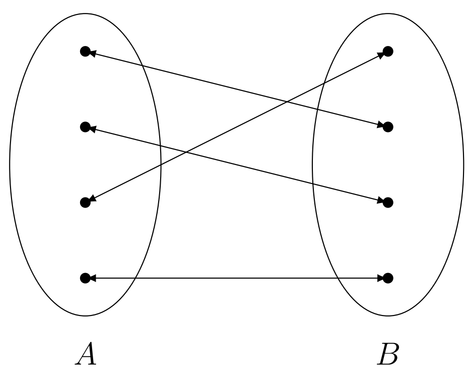

Figure 4.3: A bijective function $f : A \to B$

Notice from Figure 4.3 that $|A| = |B|$. This is very, very important, as bijections are one of the most common counting tools available.

Proving a function is bijective raises another important logical structure: **biconditional statements**, a if and only if (iff) b or $a \iff b$. Bijections are injective if and only if they are surjective. To prove an if and only if statement, first, assume a and show how b is a direct result. This is usually referred to as the forward direction. This is equivalent to proving if a then b or $a \Rightarrow b$. Second, assume b and show how a is a direct result. This is the reverse direction, or more formally, the converse of $a \Rightarrow b$. This is equivalent to proving a only if b, or $a \Leftarrow b$.

Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$. Suppose f is injective; that is, for $x, x' \in \mathbb{R}$, $f(x) = f(x')$ or $x = x'$. Then, for any $y \in \mathbb{R}$, since $y \in \mathbb{R}$, $f(y) = y$. Ergo f is surjective. Now suppose f is surjective; that is, for all $y \in \mathbb{R}$, there is an $x \in \mathbb{R}$ such that $f(x) = y$. Here $x = y$. Consider $y, y' \in \mathbb{R}$. If $f(x) = y = y' = f(x')$. Since $f(x) = y$, $x = y$. Since $f(x') = y'$, $x' = y'$. Since $y = y'$, $x = x'$, and f is injective. As f is both injective and surjective, f is bijective.

The proof above seems long and convoluted, especially for such a simple statement. There is a fine line in mathematics between concisely showing what is necessary and filling a proof with many unnecessary arguments. (This text often falls into the latter category.) As with most things, practice leads to stronger, more concise results. Audience is also important. Presented to a group of mathematicians, the above proof could possible read, "$f(x) = x$ is clearly bijective," as the audience, through experience or quick mental calculation, can see see the result immediately. The level of detail in proofs is absolutely dependent on the audience, and should be adjusted as necessary. A group of set theorists, for example, might require even more detail in the proof, while a group of children would probably want a proof by picture instead.

4.1 Exercises

Which of the following functions are injective? Surjective? Bijective?

1. $f : \mathbb{N} \to \mathbb{Z}$ defined by $f(x) = 3x$
2. $f : \mathbb{Z} \to \mathbb{N}$ defined by $f(x) = x^4$
3. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$

Section 5

Counting

There are many methods of counting. The easiest is noticing a simple a pattern, following it to its natural end, and counting the elements there. Another common argument is to establish a bijection between two sets, where the first is not immediately countable and the cardinality of the second is either already known or easily found. Both use simple multiplication rules.

5.1 Basics

Many counting problems come down to either simple addition or multiplication. With practice, more complicated problems can often be expressed in these terms.

Consider a deck cards. Suppose a person wants to draw a 10 or a face card. How many different ways can a person draw a 10 or face card? Below are all the possibilities.

$10\clubsuit$	$J\clubsuit$	$Q\clubsuit$	$K\clubsuit$	$A\clubsuit$
$10\diamondsuit$	$J\diamondsuit$	$Q\diamondsuit$	$K\diamondsuit$	$A\diamondsuit$
$10\heartsuit$	$J\heartsuit$	$Q\heartsuit$	$K\heartsuit$	$A\heartsuit$
$10\spadesuit$	$J\spadesuit$	$Q\spadesuit$	$K\spadesuit$	$A\spadesuit$

There are 20 possible ways to draw a 10 or facecard.

Theorem 17. Inclusion-Exclusion Principle For sets A and B,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Proof. $A \cup B$ has all the elements of A and all the elements of B, so, at first glance, $|A \cup B| = |A| + |B|$. However, $A \cup B$ only contains one copy of every element. (Remember, no multisets.) If $x \in A \cap B$, then x is counted twice, once in $|A|$ and once in $|B|$. Therefore, to compensate for the overcounting,

$$|A \cup B| = |A| + |B| - |A \cap B|,$$

as $A \cap B$ contains all the elements common to both. □

For easy problems, it is often assumed that A and B are **disjoint**, that is, $A \cap B = \emptyset$. In this case, $|A \cup B| = |A| + |B| - 0$. For example, if $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$, then $|A \cup B| = 6$.

This process can be generalized, although the formula is complicated. However, for disjoint sets, the formula is simple and given below.

Corollary 18. If $A_1, A_2, \ldots, A_n$ are n disjoint sets, then

$$\bigcup_{i=1}^{n} A_i = \sum_{i=1}^{n} |A_i|.$$

Proof. Since the A_i are disjoint, any intersection of any number of the A_i is 0. By the (generalized) Inclusion-Exclusion Principle, the result is immediate. □

Corollary 18 introduces some new notation, specifically

$$\begin{aligned} \bigcup_{i=1}^{n} A_i &= A_1 \cup \left(\bigcup_{i=2}^{n} A_i \right) & \sum_{i=1}^{n} |A_i| &= |A_1| + \sum_{i=2}^{n} |A_i| \\ &= A_1 \cup A_2 \cup \left(\bigcup_{i=3}^{n} A_i \right) & &= |A_1| + |A_2| + \sum_{i=3}^{n} |A_i| \\ &\vdots & &\vdots \\ &= A_1 \cup A_2 \cup \cdots \cup A_n & &= |A_1| + |A_2| + \cdots + |A_n| \end{aligned}$$

These are used to simplify notation and save space and ink. Most large symbols with a lower and upper bound can be read and expanded like this, though not all follow the same pattern. In this text, unless otherwise specified, assume it is possible to do the above expansion on such large symbols with bounds.

Theorem 19. If $A_1, A_2, \ldots A_n$ are sets such that $|A_i| = a_i$ for $1 \leq i \leq n$, then

$$|A_1 \times A_2 \times \cdots \times A_n| = a_1 a_2 \cdots a_b.$$

Proof. Consider $(x_1, x_2, \ldots, x_n) \in A_1 \times A_2 \times \cdots \times A_n$. There are a_1 choices for $x_1 \in A_1$, a_2 choices for $x_2 \in A_2$, $\ldots$, and a_n choices for $x_n \in A_n$. Since the choice of x_i does not affect the choice of x_j for $i \neq j$, we multiply each together, or

$$|A_1 \times A_2 \times \cdots \times A_n| = a_1 a_2 \cdots a_b.$$

□

Returning to the original problem of this section, Corollary 18 and Theorem 19 can be combined in several different ways to give 20. For example, there are 4 suits in the deck, each containing one 10 and four face cards, or $4(1 + 4) = 20$. Similarly, there are four tens, four jacks, four queens, four kings, and four aces, or $4+4+4+4+4 = 20$. Basic problems almost always have multiple approaches that allow for different proof styles.

5.2 Ordered Selection with Replacement

Consider a restaurant menu with three entrees a, b, and c. If a party of three goes to the restaurant, how many different combinations of orders can they make? Each person has three options, and the choice is independent of the other two. Below are listed all possible combinations:

aaa	*aab*	*aac*	*aba*	*aca*	*baa*	*caa*
bbb	*bba*	*bbc*	*bab*	*bcb*	*abb*	*cbb*
ccc	*cca*	*ccb*	*cac*	*cbc*	*acc*	*bcc*
abc	*cab*	*bca*	*bac*	*acb*	*cba*	

There are 27 total combinations for the diners. In this example, the order of choices matters: each diner is unique, so the combination *aab* is distinct from the combination *aba*. Both have two a's, but in one, the third diner selected b, while in the other, the second diner selected b. Also, replacement is allowed. All three diners can select the same meal.

Some method of proving statements involving arbitrary numbers is necessary to count the number of ways to select objects from a set with replacement.

Definition 20. **Induction** is the process of showing a statement holds for all values of r.

There are more complicated definitions and explanations of induction, but it can be done simply. To inductively prove a statement, a base case is shown for the smallest value of r. This establishes something to work with. Then the inductive step is shown by assuming that, for $r < k$, the statement is true, and for $r = k$, the statement still holds, usually by reducing it to cases with $r < k$ or showing that $r = k$ continues the statement for $r < k$.

The cardinality of $\mathcal{P}(X)$ is a perfect example of induction.

Theorem 21. Given a set X with $|X| = n$, $|\mathcal{P}(X)| = 2^n$.

Proof. Let $X = \{x_1, x_2, \ldots, x_n\}$. If $n = 1$, then $\mathcal{P}(X) = \{\emptyset, \{x_1\}\}$, so $|\mathcal{P}(X)| = 2$. Assume that, for $n < k$, $|\mathcal{P}(X)| = 2^n$ and consider $n = k$. We know $|\mathcal{P}(x_1, x_2, \ldots, x_{k-1})| = 2^{k-1}$, and must now consider $|\mathcal{P}(x_1, x_1, x_2, \ldots, x_{k-1}, x_k)|$. We have have added one element x_k to the set. For any subset $X' \subseteq X$, either $x_k \in X'$ or $x_k \notin X'$, so for every subset of $\{x_1, x_2, \ldots, x_{k-1}\}$, we now have two: one without x_k and one with. So $|\mathcal{P}(x_1, x_1, x_2, \ldots, x_{k-1}, x_k)| = 2 \cdot 2^{k-1} = 2^k$. □

Theorem 22. The number of ways to choose r objects from a set of size n where order matters is n^r.

Proof. Let X be a set such that $|X| = n$. Suppose $r = 1$. There are $n = n^1$ way to select 1 object from X. Assume that, for $r < k$ with $|X| = n$, there are n^r ways to choose r objects from X, and consider $r = k$. We know that the number of ways of selecting $k - 1$ objects from X is n^{k-1}. There are n

choices for the kth selection, so there are $n^{k-1} \cdot n = n^k$ ways to choose k objects from a set of size n. □

n	n	n	$\cdots$	n	n	n
1	2	3	$\cdots$	$r-2$	$r-1$	r

Figure 5.1: There are r boxes, each with n choices.

Returning to the original problem of this section, there were three diners ordering from a menu with three options. Using Theorem 22, there are then $3^3 = 27$ choices, which was the result obtained by writing out all 27 options. Which method is faster?

5.3 Ordered Selection without Replacement

Consider a bag of four marbles a, b, c, and d. How many different ways can three people each get one marble? The first people has four choices, the second only three, and the third only two. Below are listed all possible combinations:

abc	*abd*	*acb*	*acd*	*adb*	*adc*
bac	*bad*	*bca*	*bcd*	*bda*	*bdc*
cab	*cad*	*cba*	*cbd*	*cda*	*cdb*
dab	*dac*	*dba*	*dbc*	*dca*	*dcb*

There are 24 possible combinations of marbles. Once again, order matters. *abc* is different than *cab*. In one, the first person has marble a, while in the other, the second person has marble a. Here, replacement is not allowed. Two people cannot have the same marble.

Writing out all of the multiplication involved in this counting is possible, but lengthy.

Definition 23. A **factorial**, $n!$, is the product $n(n-1)(n-2)\ldots(2)(1)$. $0!$ is defined to be 1.

Factorials come from $\Gamma(x)$, a complicated function with any domain. However, as combinatorists prefer to count things, restricting $\Gamma(X)$ to $\mathbb{N} \cup \{0\}$ gives something discrete and tangible. Very rarely does a combinatorist use negative numbers for counting, and, unfortunately, the amazing results that come from inserting negative numbers into common combinatorial functions are much beyond the scope of this text. [1]

$$\begin{aligned}
0! &= 1 && = 1 \\
1! &= 1 && = 1 \\
2! &= 2 \cdot 1 && = 2 \\
3! &= 3 \cdot 2! && = 6 \\
4! &= 4 \cdot 4! && = 24 \\
5! &= 5 \cdot 4! && = 120 \\
6! &= 6 \cdot 5! && = 720 \\
7! &= 7 \cdot 6! && = 5040 \\
&\vdots && \vdots \\
n! &= n \cdot (n-1)! && = n(n-1)(n-2)\dots(2)(1)
\end{aligned}$$

Factorials are recursively defined, that is, $n! = n \cdot (n-1)!$, provided $n \geq 1$. (To prove this, simply continue to reapply the recursive definition until arriving at the factorial formula. Induction may be helpful.) They also present a very common pattern. Like many things combinatorics, knowing the first several factorials is very important. The pattern appears quite often.

Theorem 24. If $n \geq r$, the number of ways to choose r elements from a set of size n without replacement is

$$n(n-1)\cdots(n-r+2)(n-r+1) = \frac{n!}{(n-r)!}.$$

[1]For one of many fascinating examples of negative numbers used to count something, Richard Stanley proved that the number of acyclic orientations of a graph is $(-1)^n\chi_G(-1)$, where $\chi_G(k)$ is the chromatic polynomial of the graph. It may seem like jibberish, but it's a terrific theorem that stretches the boundaries of combinatorics. Most other results follow a similar trend: take a known combinatorial function, plug in negative numbers, and try to figure out if something important is counted.

Proof. Let X be a set such that $|X| = n$. If $r = 1$, then there are n choices, or

$$n = \frac{n(n-1)(n-2)\cdots(2)(1)}{(n-1)(n-2)\cdots(2)(1)} = \frac{n!}{(n-1)!}.$$

Assume, for $r < k$, there are $n!/(n-r)!$ ways to choose r elements from a set of size n, and consider $r = k$. There are $n(n-1)\cdots(n-(k-1)+2)(n-(k-1)+1)$ ways to choose the first $k-1$ elements, leaving $n-(k-1) = n-k+1$ elements to choose for the kth selection. So there are

$$\begin{aligned} &n(n-1)\cdots(n-(k-1)+2)(n-(k-1)+1)(n-(k-1)) \\ &= n(n-2)(n-3)\cdots(n-k+3)(n-k+2)(n-k+1) \\ &= \frac{n!}{(n-k)!} \end{aligned}$$

ways to choose k elements from a set of size n. □

n	$n-1$	$n-2$	$\cdots$	$n-r+3$	$n-r+2$	$n-r+1$
1	2	3	$\cdots$	$r-2$	$r-1$	r

Figure 5.2: There are r boxes, each with a decreasing number of choices.

Returning to the original problem of the section, there were three people choosing from four marbles. Using Theorem 24, there are $4!/4-3! = 24$ ways to choose. Which is faster?

5.4 Unordered Selection without Replacement

Consider a box of items at a store containing a, b, c, d, and e. Suppose a single person goes in with the intention of buying three things from the box. How many different combinations of items can the person get? Below are listed all possible combinations:

$$\begin{array}{cccccc} abc & abd & abe & acd & ace & ade \\ bcd & bce & bde & & & \\ cde & & & & & \end{array}$$

There are 10 possible combinations. Order no longer matters, as a single person gets all the items. The pattern *abc* is the same as *cba*, since the person will walk out of the store with items a, b, and c in both cases.

Before counting the number of selections, overcounting and coping with overcounting must be mentioned. Simple formulas for combinatorics often overcount. It's a fact of the field. It's also simple to correct. Overcount by 1? Subtract 1 from the total. Count everything 7 times? Divide the total by 7. Examples can also be useful to check formulas. Working with arbitrary sets is rarely easy to just talk about, so pictures and discrete formulas are very helpful. Using those, was the count too big? Compensate.

Theorem 25. The number of ways to choose r elements from a set of size n where order does not matter is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Proof. Let X be a set of size n. From Theorem 24, we know there are $n!/(n-r)!$ ways to select r elements from X where order matters. However, this overcounts, as it places an order on the selection. We must therefore divide out the total number of orders. Using Theorem 24, we know there are $r!/(r-r)! = r!$ ways to order these r elements, as there are r positions, each taken without replacement from a set of size r. So there are

$$\frac{\frac{n!}{(n-r)!}}{r!} = \frac{n!}{r!(n-r)!} = \binom{n}{r}$$

ways to select r elements from a set of size n. □

5.5 Exercises

1. a) Suppose Barbara and Jane want to go to the theater. There are five movies Barbara wants to see. Jane has no interest in any of those five, but wants to see three others. However, they're good friends and will go see something together. How many different movies do they want to see?

b) Suppose Chad is going to tag along. He doesn't want to see anything they want to see, and instead wants to go to one of two foreign films playing. However, once again, they'll all go together to see something. How many different movies do all three want to go to?

2. An ice cream place has four flavors, five toppings, three syrups, and six sizes. How many different orders could a customer make?

3. Richard has two brothers and three sisters. He wants to order birthday cards for the year in bulk to save money. After some research, he finds a website with 30 cards for men and 54 cards for women. How many different sets of five cards can Richard order?

4. You are given the letters a, b, c, d, and e. How many...

   a) strings of length 5 can you make?
   b) strings of length 4 can you make?
   c) strings of length 3 can you make?
   d) strings of length 2 can you make?
   e) strings of length 1 can you make?
   f) strings of any length can you make?

5. Jane has ten good friends. She recently won three tickets for a cruise and wants to take two friends with her. How many different selections can she make?

Section 6

Graph Theory

Graph theory is one of many fields in combinatorics. "Classical" graph theory was essentially finished in the 20th century. Results pertaining to planar graphs, graph coloring, existence and isomorphisms, flow and capacity, edge or vertex optimization, and other introductory problems have all be proved, reproved, and disseminated to a wide audience. New research is still happening, but, like many other fields, it's on the fringes, attacking special cases or specific problems, or either extending graph theoretic concepts to new fields or using other fields to reexamine graph theoretic concepts.

After some terminology, graph theory presents a great application of the counting concepts just learned. Seeing combinatorial principles in action is a huge part of learning combinatorics, as well as reapplying the principles to new combinatorial objects.

Definition 26. A **graph** $G = (V, E)$ is an order pair containing a vertex set V and an edge set E. A **vertex** is a node or point in a graph. An **edge** is a line with two vertices (not necessarily distinct) as endpoints.

The loose definition of edges can be problematic. A **loop** is an edge connected to a single vertex. **Parallel edges** are edges who share the same endpoints. Both of these things, more often than not, cause problems in graph theory. It is generally better to assume the graph is **simple**, that is, the graph contains no loops or parallel edges.

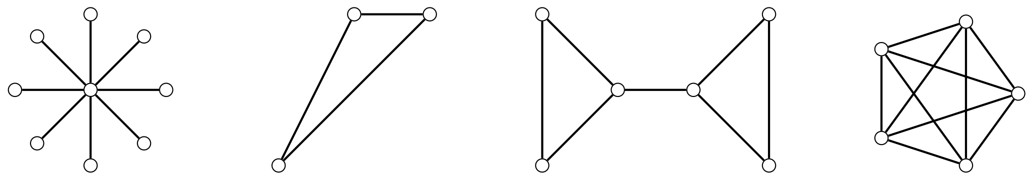

Figure 6.1: Several graphs

Definition 27. The **degree** of a vertex, $d(v)$ for $v \in V$ is the number of edges connected to it.

Definition 28. The **complete graph** K_n on n vertices is the graph $G(V, E)$ where $|V| = n$ and every vertex shares an edge with every other vertex.

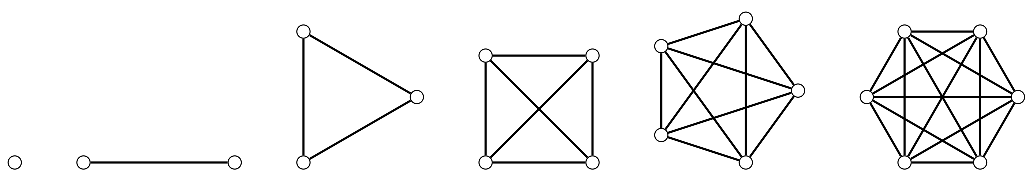

Figure 6.2: K_1, K_2, K_3, K_4, K_5, and K_6

Theorem 29. K_n has exactly $\binom{n}{2}$ edges.

Proof. We know every vertex shares an edge with every other vertex. If we think of edges as a string of length, its endpoints uv, then there are $\binom{n}{2}$ ways to create such a string by Theorem 25, since $|V| = n$. □

K_n contains every possible edge, so $\binom{n}{2}$ is an upper bound for $|E|$ in any other graph. While the edges in a simple graph depend on the number of vertices, there can be up to an infinite number of vertices. Infinite graphs are an interesting area of research, but very difficult to ponder.

Definition 30. A **bipartite graph** $G = (X, Y, E)$ is a graph with disjoint vertex sets X and Y and edge E such that the only edges in the graph go from X to Y, or, if $e \in E$, then $e = xy$ for $x \in X$ and $y \in Y$.

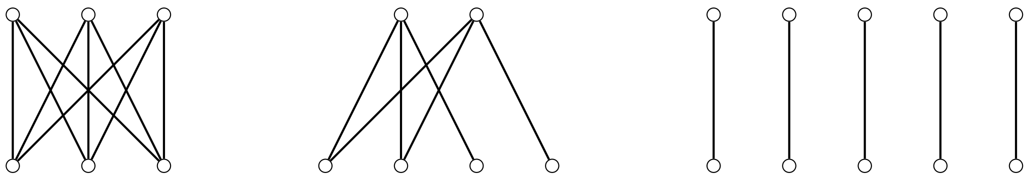

Figure 6.3: Several bipartite graphs

The **complete bipartite graph** $K_{m,n}$ is a natural extension of K_n. It is the graph $G = (X, Y, E)$ with $|X| = m$, $|Y| = n$, and all possible edges between X and Y.

Theorem 31. $K_{m,n}$ has mn edges.

Proof. Once again, an edge is a string of length two. As there are m options for the first position and n for the second, by Theorem 19, there are mn edges. □

Aside from the number of edges, a very clever and simple theorem is as follows:

Theorem 32. Let $G = (V, E)$ be a graph. The sum of the degrees of the vertices in V is twice the number of edges, or

$$\sum_{v \in V} d(v) = 2|E|.$$

Proof. Every edge contains two endpoints and therefore contributes to two degrees, which yields the desired result. □

This is often referred to as the "Handshaking" lemma or theorem. It was one of the very first results in graph theory, proved during Euler's career.

Section 7

What Next?

Over the last century, combinatorics and its applications have become a major field in mathematics. The counting techniques presented here are just the beginning. The short introduction to graph theory illustrates how easy it is to use these methods to create something larger and more complicated. However, from here, more theory is needed. Most combinatorics today involves a good amount of abstract algebra. New research often uses results from analysis and linear algebra. Using very advanced techniques, for example, can lead to a generalization of the Inclusion/Exclusion principle to grander combinatorial objects.

7.1 Recommended Reading

These books are recommended for further research. Each is accompanied by an explanation of why the book is important or useful.

- Ian Stewart, *Concepts of Modern Mathematics*: Stewart wrote this book as a companion to a course he taught in the 70s and 80s, introducing non-mathematicians to real math. It covers many of the things mentioned here in greater detail. It's also written to be read by non-mathematicians, so it's more approachable than some broad math introductions. The original is now out of print, but Dover has republished it since.

- Daniel Velleman, *How to Prove It: A Structured Approach*: Proofs are the backbone of math. Velleman's book requires little math knowledge and covers the important details of proof writing. Good math programs include some class covering proofs. This is one of the many texts used for such courses. The book is available through Cambridge University Press.

- John Fraleigh, *A First Course in Abstract Algebra*: Abstract, or modern, algebra is incredibly important to combinatorics. Groups, rings, and fields play a large role in defining and using more advanced combinatorial objects. However, a good introductory textbook is difficult to find. Fraleigh's work covers all the pertinent material and is written a level that can, for the most part, be easily understood. There are multiple editions. All cover at least the basics, and several cover more advanced topics. The 7th edition is available through Pearson.

- Richard Stanley, *Enumerative Combinatorics*: This is one of the most definitive combinatorics textbooks. While targeted to an advanced audience, it covers almost everything a new combinatorist needs to know. A solid grasp of algebra and proof technique is required to make it through the book. The book is available through Cambridge University Press. Volume 1 can be downloaded for free from his website, http://math.mit.edu/~rstan/ec/, along with various errata and addenda.

7.2 Software and Computers

Like most mathematics, combinatorics often relies heavily on computers. The Four Color Theorem, which essentially states a map needs no more than four colors, was originally proved by reducing the problem to a finite number of cases, then running all the possibilities through a computer. Since then, computers have come quite a long way and are used in similar circumstances quite often now.

Mathematicians need a way to write mathematics. LaTeX, http://www.latex-project.org/, is the industry standard. Newer office software, like Microsoft Office or Open Office, are capable of complicated manuscripts, but LaTeXis built completely for this purpose. LaTeXis a computer language, similar to a programming language. A document is coded, run through a compiler, and turned into a variety of formats. The

LaTeX Wikibook, http://en.wikibooks.org/wiki/LaTeX, is a great introduction to the topic. There are many other easily found resources on the internet. Texmaker, http://www.xm1math.net/texmaker/; TeXworks, http://www.tug.org/texworks/; and LyX, http://www.lyx.org/, are common complete editors available to all operating systems. However, using a text editor like Emacs or Notepad++, then compiling via the command line is just as common.

Sage, http://www.sagemath.org/, is open-source software used for abstract computation. Built in Python, it is an invaluable time-saver. There are currently no solid introductions to the software, but the tutorial on Sage's website, http://www.sagemath.org/doc/tutorial/index.html, provides the basics. A great feature of Sage is the ability to use (older versions of) Sage on the internet at http://www.sagenb.org/. Mathematica and Maple are equivalent to Sage in many aspects, and better in many others, but come with an incredibly large price tag. Software, and programming as well, should only be used to simplify the process once it's completely understood; i.e. running everything through a computer without understanding it perpetuates a lack of understanding.

Appendix A

Exercise Answers

A.1 Sets

1. a) $|X|$: 1
   $\mathcal{P}(X)$: $\{\emptyset, \{1\}\}$
   $|\mathcal{P}(X)|$: 2

   b) $|X|$: 2
   $\mathcal{P}(X)$: $\{\emptyset, \{5\}, \{7\}, \{5, 7\}\}$
   $|\mathcal{P}(X)|$: 4

   c) $|X|$: 3
   $\mathcal{P}(X)$: $\{\emptyset, \{\pi\}, \{e\}, \{i\}, \{\pi, e\}, \{\pi, i\}, \{e, i\}, \{\pi, e, i\}\}$
   $|\mathcal{P}(X)|$: 8

   d) $|X|$: 4
   $\mathcal{P}(X)$: $\{\emptyset, \{10\}, \{100\}, \{1000\}, \{10000\}, \{10, 100\}, \{10, 1000\},$
   $\{10, 10000\}, \{100, 1000\}, \{100, 10000\}, \{1000, 10000\}$
   $, \{10, 100, 1000\}, \{10, 100, 10000\}, \{10, 1000, 10000\},$
   $\{100, 1000, 10000\}, \{10, 100, 1000, 10000\}\}$
   $|\mathcal{P}(X)|$: 16

2. a) $A = \{a\}$, $B = \{1, 2, 3\}$

   $$A \times B = \{(a, 1), (a, 2), (a, 3)\}$$

   b) $A = \{\text{blue, green}\}$, $B = \{5, 6\}$

   $$A \times B = \{(\text{blue}, 5), (\text{blue}, 6), (\text{green}, 5), (\text{green}, 6)\}$$

c) $A = \{\pi, e, i\}$, $B = \{a, b, c\}$, $C = \{\heartsuit, \spadesuit\}$

$$\begin{aligned} A \times B \times C = \{&(\pi, a, \heartsuit), (\pi, b, \heartsuit), (\pi, c, \heartsuit), (\pi, a, \spadesuit), (\pi, b, \spadesuit), \\ &(\pi, c, \spadesuit), (e, a, \heartsuit), (e, b, \heartsuit), (e, c, \heartsuit), (e, a, \spadesuit), \\ &(e, b, \spadesuit), (e, c, \spadesuit), (i, a, \heartsuit), (i, b, \heartsuit), (i, c, \heartsuit), \\ &(i, a, \spadesuit), (i, b, \spadesuit), (i, c, \spadesuit)\} \end{aligned}$$

A.2 Important sets

There are many options for each question. One is provided.

1. All even integers

   $\{n | \text{for } t \in \mathbb{Z}, n = 2t\}$

2. All odd integers

   $\{x | \text{for } z \in \mathbb{Z}, x = 2z + 1\}$

3. Integers over powers of 2.

   $\{x | \text{for } z \in \mathbb{Z}, n \in \mathbb{N} \cup \{0\}, x = \frac{z}{2^n}\}$

4. Negative numbers.

   $\{x \in \mathbb{R} | x < 0\}$

A.3 Functions

1. $f : \mathbb{N} \to \mathbb{Z}$ defined by $f(x) = 3x$

   This function is injective, as, if for any $n, n' \in \mathbb{N}$, $f(n) = 3n = 3n' = f(n')$, then $n = n'$. f is not surjective, as, for example, $-1 \in \mathbb{Z}$ does not come from any element of the domain.

2. $f : \mathbb{Z} \to \mathbb{N} \cup \{0\}$ defined by $f(x) = x^4$

   f is neither surjective nor injective. f is not injective as, for example, $f(1) = 1 = 1 = f(-1)$ but $1 \neq -1$. f is not surjective, as, for example, as $f(x) = 2$ or $x = \sqrt[4]{2}$ is not in the domain.

3. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$

   f is injective as, if for any $x, x' \in \mathbb{R}$, $f(x) = x^3 = (x')^3 = f(x')$, then $\sqrt[3]{x^3} = x = x' = \sqrt[3]{(x')^3}$. f is surjective, as, for any $y \in \mathbb{R}$, if $f(x) = y$ or $x^3 = y$, then $\sqrt[3]{x^3} = x = \sqrt[3]{y} \in \mathbb{R}$. As f is injective and surjective, f is bijective.

A.4 Counting

1. a) Suppose Barbara and Jane want to go to the theater. There are five movies Barbara wants to see. Jane has no interest in any of those five, but wants to see three others. However, they're good friends and will go see something together. How many different movies do they want to see?

   Barbara wants to see 5, Jane 3, so together, using Corollary 18, they want to see 8 total.

   b) Suppose Chad is going to tag along. He doesn't want to see anything they want to see, and instead wants to go to one of two foreign films playing. However, once again, they'll all go together to see something. How many different movies do all three want to go to?

   Again, Barbara 5, Jane 3, Chad 2, so by Corollary 18, 10 total.

2. An ice cream place has four flavors, five toppings, three syrups, and six sizes. How many different orders could a customer make?

   Using Theorem 19, there are $4 \cdot 5 \cdot 3 \cdot 6$ total orders.

3. Richard has five brothers and three sisters. He wants to order birthday cards for the year in bulk to save money. After some research, he finds a website with 30 cards for men and 54 cards for women. How many different sets of five cards can Richard order?

   Richard has 30 choices for each brother and 54 for each sister. Using Theorems 19 and 22, he has $(30)^5(54)^2$ total choices.

4. You are given the letters a, b, c, d, and e. How many...

   a) strings of length 5 can you make?

By Theorem 24, we have

$$\frac{5!}{(5-5)!} = 5! = 120$$

possible strings of length 5.

b) strings of length 4 can you make?

By Theorem 24, we have

$$\frac{5!}{(5-4)!} = 5 \cdot 4 \cdot 3 \cdot 2 = 120$$

possible strings of length 4.

c) strings of length 3 can you make?

By Theorem 24, we have

$$\frac{5!}{(5-3)!} = 5 \cdot 4 \cdot 3 = 60$$

possible strings of length 3.

d) strings of length 2 can you make?

By Theorem 24, we have

$$\frac{5!}{(5-2)!} = 5 \cdot 4 = 20$$

possible strings of length 2.

e) strings of length 1 can you make?

By Theorem 24, we have

$$\frac{5!}{(5-1)!} = 5$$

possible strings of length 1.

f) strings of any length longer than 0 can you make?

By Corollary 18, we have $5 + 20 + 60 + 120 + 120 = 325$ possible strings of any length.

5. Jane has ten good friends. She recently won three tickets for a cruise and wants to take two friends with her. How many different selections can she make?

   Jane has ten friends and wants to pick two. Order does not matter, so by Theorem 25, there are

$$\binom{10}{2} = \frac{10!}{2!(10-2)!} = \frac{10 \cdot 9}{2} = 45$$

   possible ways for her to select friends.

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