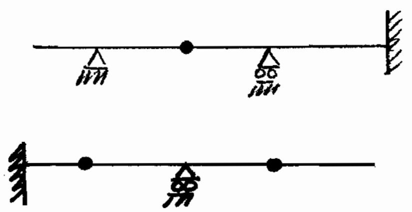
**CE 461 – Structural Analysis** **Name:** C.R. Bennett
**EXAM No. 3**

(2) 1. Given the following "real" beam, draw the corresponding conjugate beam.



(2) 2. What is/are the *unknown(s)* in the method of consistent deformations?

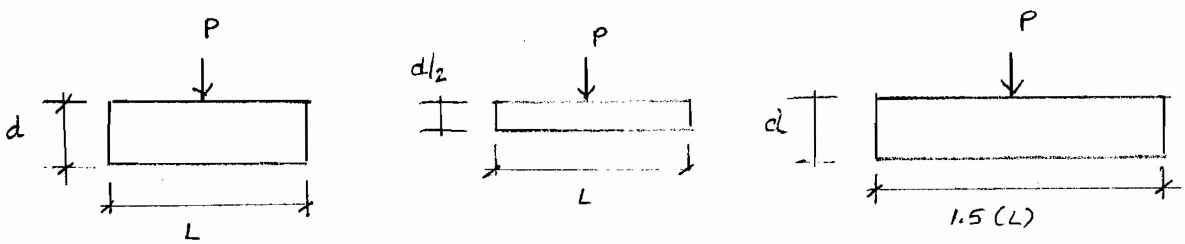
a. Work
b. Energy
c. Forces
d. Displacements
e. Equation(s) of Condition
f. I don't know, don't ask me such crazy questions

(3) 3. What does Maxwell's Law of Reciprocal Deflections state? A picture may be helpful, but will not be sufficient by itself.

Deflection @ a pt. "i" due to a unit load at a pt. "j" is equal to deflection at "j" due to a unit load at "i".

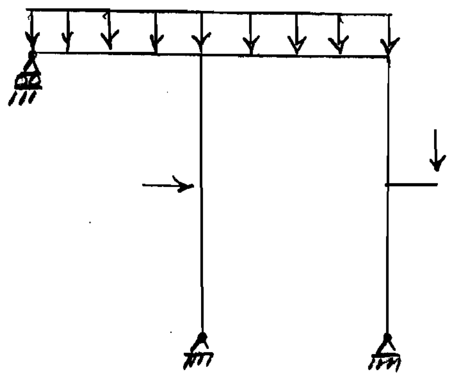

$$f_{ji} = f_{ij}$$

(2) 4. In which of the following cases would it be most important to account for shear deformations when calculating the total deflection of the beam?




(a) short & deep span. (b) (c)

(4) 5. Given the following structure, how many simultaneous equations would be required to solve for the redundant reactions if you were using the force method?



5 UNK > 3 EQNS

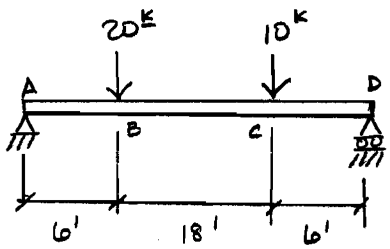
2° ind.

2 eqns. needed to solve for redundandants

(2) 6. What are two other names for the force method?

- Method of consistent deformations
- Flexibility method

(5) 7. If the two loads shown on the beam below are applied simultaneously, the external work is 39 in-kips. If the two loads are instead applied sequentially, first the 20 kip load, then the 10 kip load, what is the total internal strain energy stored in the beam after both loads are applied?

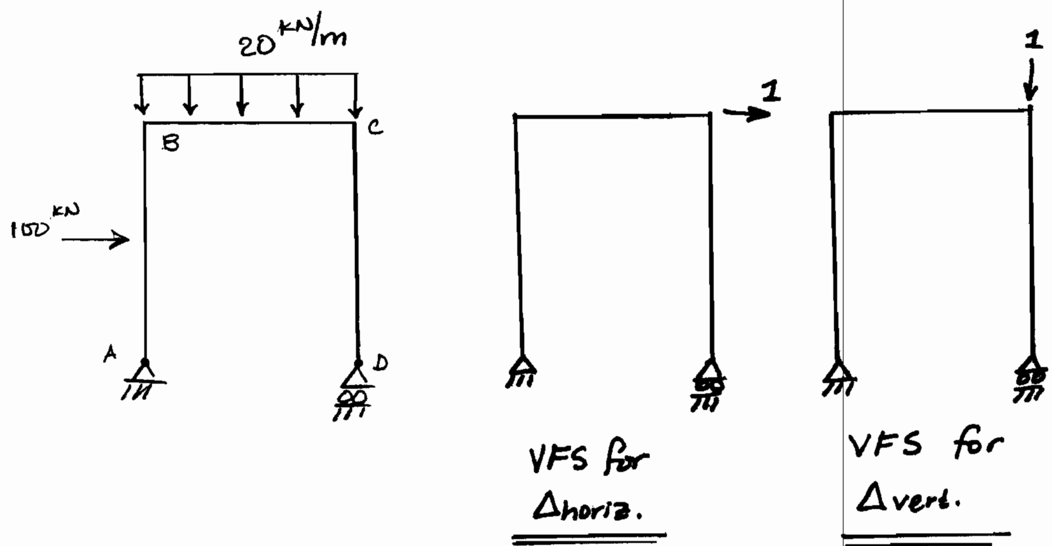



$\mathcal{U}_e = \mathcal{U}_i$

$\therefore \mathcal{U}_i = 39''\text{-}k$

(3) 8. Draw the qualitative deflected shape for the following loaded beam.

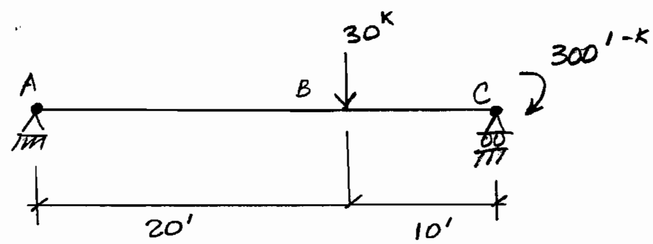
(4) 9. Draw the virtual force systems (VFS's) you would use if you were going to solve for horizontal and vertical deflection at joint C of the following frame using the method of virtual work.



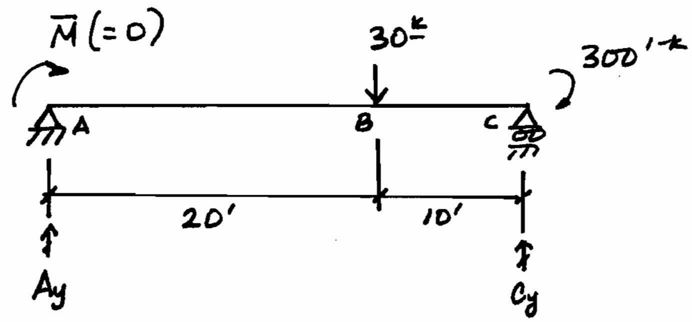

(3) 10. Name three types of loadings that cause stresses in indeterminate structures, but not in determinate structures.

- Temperature
- differential settlement
- fabrication / construction errors.

(25) 11. Using either Castigliano's Second Theorem or the Method of Virtual Work, determine the rotation at point A of the beam. EI is constant. (you may neglect shear deformations)




## Castigliano's 2nd Theorem Approach:



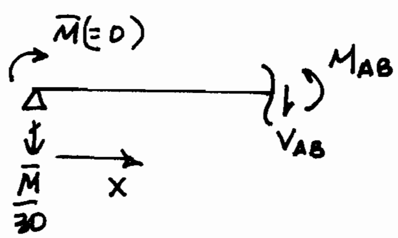

Rxns: $+\uparrow\Sigma F_y = 0 = A_y - 30 + C_y$

$\curvearrowleft + \Sigma M_A = 0 = \bar{M} + 30^k(20') - C_y(30') + 300'^{-k}$

$$\underline{C_y = \frac{\bar{M}}{30} + 30}$$

$$\underline{A_y = -\frac{\bar{M}}{30}}$$

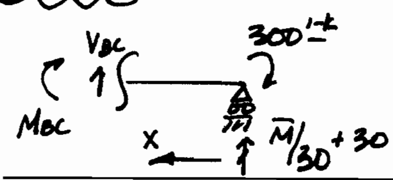
Region AB




$+\curvearrowright \Sigma M_{AB} = 0 = M_{AB} + \frac{\bar{M}}{30}(x) - \bar{M}$

$$\underline{M_{AB} = -\frac{\bar{M}}{30}(x) + \bar{M}}$$

Region BC




$\curvearrowleft + \Sigma M_{BC} = 0 = 300'^{-k} - \left(\frac{\bar{M}}{30} + 30\right)(x) + M_{BC}$

$$\underline{M_{BC} = -300 + \left(\left(\frac{\bar{M}}{30}\right) + 30\right)x}$$

| Region | Limits | Origin | $M(x)$ | $\partial M / \partial \bar{M}$ |
|---|---|---|---|---|
| AB | 0 - 20 | A | $-\frac{\bar{M}}{30}(x) + \bar{M}$ | $\left(1 - \frac{x}{30}\right)$ |
| BC | 0 - 10 | C | $-300 + \left(\frac{\bar{M}}{30} + 30\right)x$ | $\left(\frac{x}{30}\right)$ |

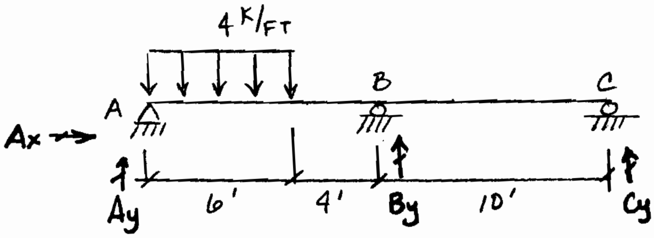
$$\theta = \int_0^L \left(\frac{\partial M}{\partial \bar{M}}\right)\frac{M}{EI}\, dx$$

$$\theta = \int_0^{20} \left(1 - \frac{x}{30}\right)\left(\cancelto{0}{-\frac{\bar{M}}{30}x + \bar{M}}\right) + \int_0^{10} \left(\frac{x}{30}\right)\left(-300 + \cancelto{0}{\frac{\bar{M}}{30}x} + 30x\right) dx$$

$$\theta = \int_0^{10} \left(\frac{1}{30}x\right)\left(-300 + 30x\right) dx$$

$$\boxed{\theta = -\frac{166.67}{EI}, \text{ neg } \therefore}$$

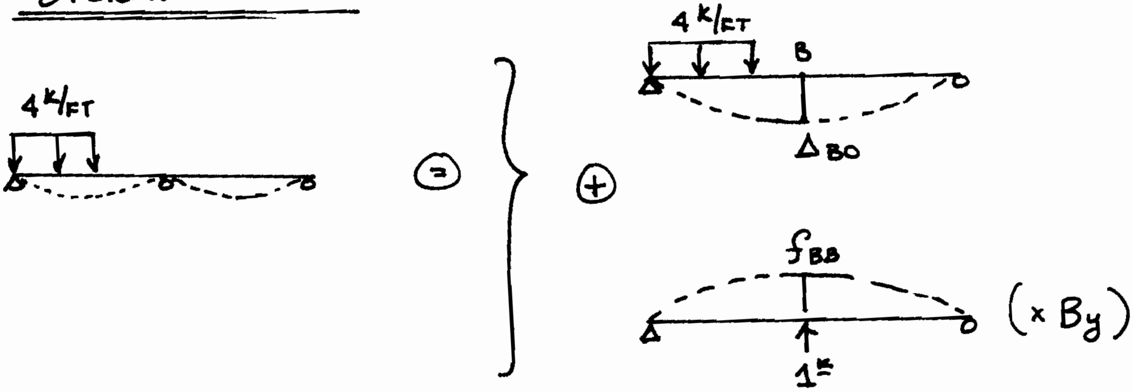
(25) 12. Determine all support reactions for the following beam. If you wish, you may use generic "canned" equations (attached at the end of this test) as part of your solution.




1° indeterminate.

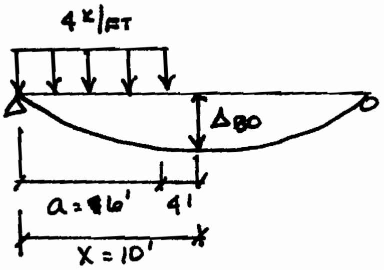
I chose $B_y$ as the redur

OVERALL APPROACH:




COMPATIBILITY EQN: $0 = \Delta_{BO} + f_{BB} B_y$

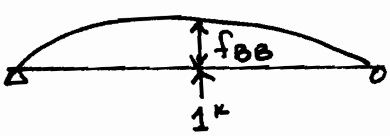
SOLVE FOR DEFLECTIONS OF PRIMARY BEAM: $(\Delta_{BO}$ & $f_{BB})$




$$\Delta_{BO} = y(10') = -\frac{wa^2}{24EIL}(L-x)(-2x^2 + 4Lx - a^2)$$

$$\Delta_{BO} = \frac{-4(6)^2}{24EI(20)}(20-10)(-2(10)^2 + 4(20)(10) - 6^2)$$

$$\Delta_{BO} = \frac{-1692}{EI}$$




$$f_{BB} = \frac{PL^3}{48EI} = \frac{1(20)^3}{48EI} = \frac{166.7}{EI}$$

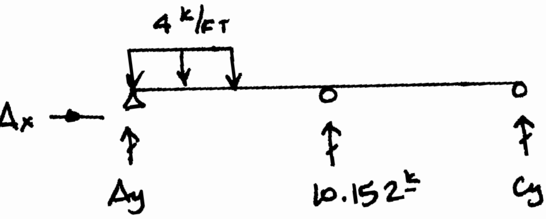
## Solve for Redundant Rxn:

$$0 = \Delta_{BO} + f_{BB} B_y$$

$$0 = \frac{-1692}{EI} + \frac{166.7}{EI}(B_y)$$

$$\underline{B_y = 10.152^k \uparrow}$$

## Solve for Remaining Rxns:



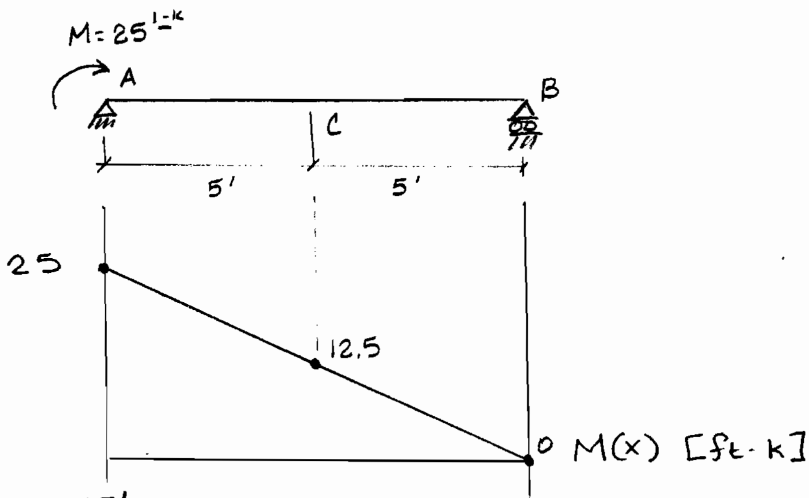

$$+\uparrow \Sigma F_y = 0 = -4(6) + A_y + C_y -$$

$$\circlearrowleft + \Sigma M_A = 0 = -4(6)(3) + 10.15$$

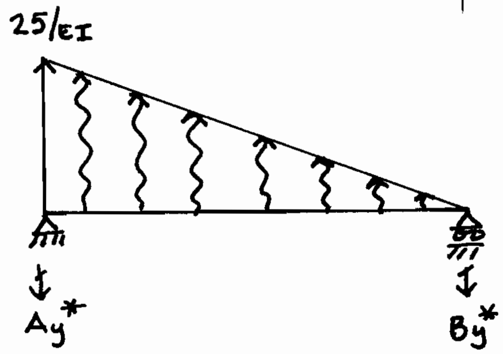
$$\underline{C_y = 1.476^k \downarrow}$$

$$\underline{A_y = 15.324^k \uparrow}$$

(20) 13. Given the following beam and its moment diagram, determine the slope and deflection at point C. Use the conjugate beam method. EI is constant.




Conjugate beam under conjugate loading



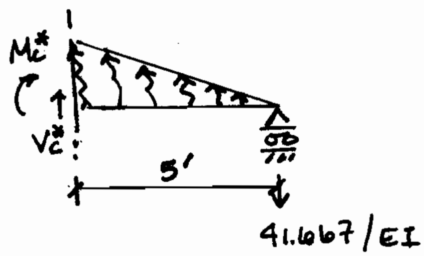

C.B. Rxns:

$+\uparrow \Sigma F_y = 0 = \frac{25}{EI}\left(\frac{1}{2}\right)(10') - A_y^* - B_y^*$

$\circlearrowright + \Sigma M_A = 0 = \frac{25}{EI}\left(\frac{1}{2}\right)(10)\left(\frac{10}{3}\right) - 10B_y^*$

$B_y^* = \frac{41.667}{EI} \therefore \downarrow$

Solve for $M_c^*$ & $V_c^*$




$+\uparrow \Sigma F_y = 0 = \frac{1}{2}\left(\frac{12.5}{EI}\right)(5) + V_c^* - 41.667/EI$

$V_c^* = \theta_c = \frac{10.42}{EI}$ Rad, pos $\therefore \measuredangle$

$\circlearrowright + \Sigma M_c^* = M_c^* - \frac{1}{2}\left(\frac{12.5}{EI}\right)(5)\left(\frac{5}{3}\right) + \frac{41.667}{EI}(5)$

$M_c^* = y_c^* = -\frac{156.25}{EI}$, neg $\therefore \downarrow$

**Internal Work (Strain Energy) for Axial Forces** $U_i = \dfrac{N^2 L}{2AE}$

**Internal Work (Strain Energy) for Bending** $U_i = \displaystyle\int_0^L \frac{M^2 dx}{2EI}$

**Internal Work (Strain Energy) for Shear** $U_i = \displaystyle\int_0^L \left(\frac{V^2}{2GA_v}\right)dx$, where $A_v = A/K$

K = 1.2 for rectangles
K =10/9 for circular cross-sections
K = approx. 1 for I-beams, where A is the area of the web
K = 2 for tubes

**Method of Virtual Work Applied to Trusses** $1(\Delta) = \sum n_v \left(\dfrac{NL}{AE}\right)$

**Method of Virtual Work Applied to Beams**

$$\begin{cases} 1(\Delta) = \left[\displaystyle\int_0^L \frac{m_v M}{EI} dx\right] + \left[\displaystyle\int_0^L \frac{v_v V}{GA_v} dx\right] \\ 1(\theta) = \left[\displaystyle\int_0^L \frac{m_v M}{EI} dx\right] + \left[\displaystyle\int_0^L \frac{v_v V}{GA_v} dx\right] \end{cases}$$

**Method of Virtual Work Applied to Frames**

$$\begin{cases} 1(\Delta) = \left[\sum n_v \left(\dfrac{NL}{AE}\right)\right] + \left[\sum \displaystyle\int_0^L \frac{m_v M}{EI} dx\right] + \left[\sum \displaystyle\int_0^L \frac{v_v V}{GA_v} dx\right] \\ 1(\theta) = \left[\sum n_v \left(\dfrac{NL}{AE}\right)\right] + \left[\sum \displaystyle\int_0^L \frac{m_v M}{EI} dx\right] + \left[\sum \displaystyle\int_0^L \frac{v_v V}{GA_v} dx\right] \end{cases}$$

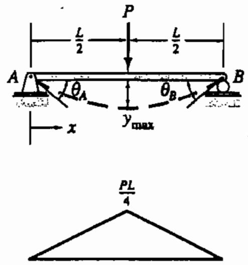
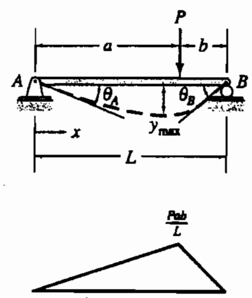
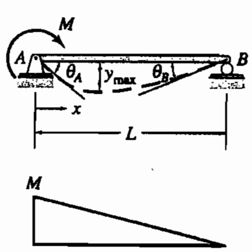
**Castigliano's Second Theorem Applied to Trusses** $\Delta = \sum \left(\dfrac{\partial N}{\partial P}\right)\dfrac{NL}{AE}$

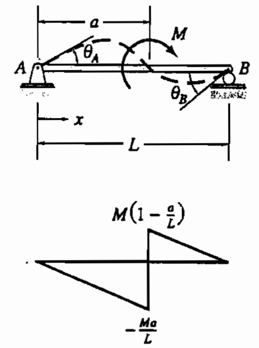
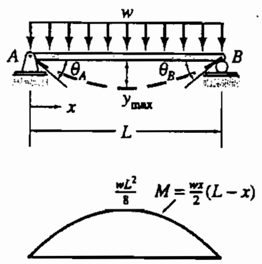
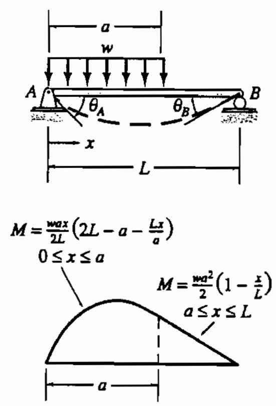
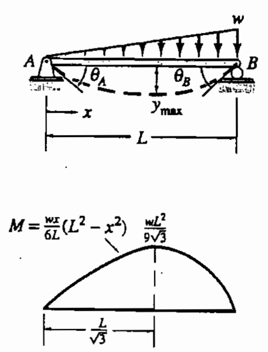
**Castigliano's Second Theorem Applied to Beams**

$$\begin{cases} \Delta = \left[\displaystyle\int_0^L \left(\frac{\partial M}{\partial P}\right)\frac{M}{EI} dx\right] + \left[\displaystyle\int_0^L \left(\frac{\partial V}{\partial P}\right)\frac{V}{GA_v} dx\right] \\ \theta = \left[\displaystyle\int_0^L \left(\frac{\partial M}{\partial \bar{M}}\right)\frac{M}{EI} dx\right] + \left[\displaystyle\int_0^L \left(\frac{\partial V}{\partial \bar{M}}\right)\frac{V}{GA_v} dx\right] \end{cases}$$

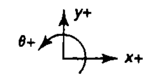
**Castigliano's Second Theorem Applied to Frames**

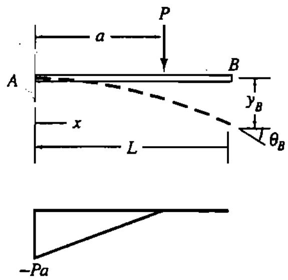
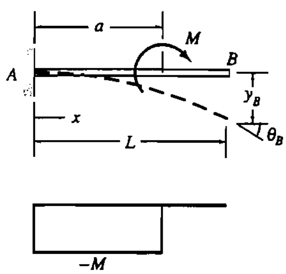
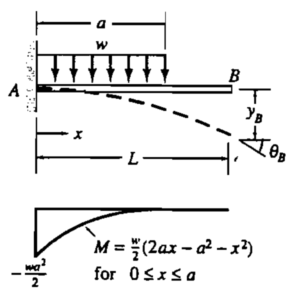
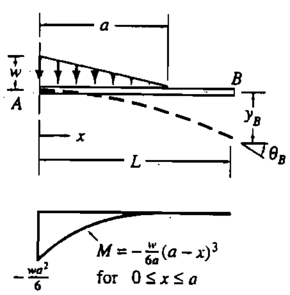
$$\begin{cases} \Delta = \left[\sum \left(\dfrac{\partial N}{\partial P}\right)\dfrac{NL}{AE}\right] + \left[\sum \displaystyle\int_0^L \left(\frac{\partial M}{\partial P}\right)\frac{M}{EI} dx\right] + \left[\displaystyle\int_0^L \left(\frac{\partial V}{\partial P}\right)\frac{V}{GA_v} dx\right] \\ \theta = \left[\sum \left(\dfrac{\partial N}{\partial \bar{M}}\right)\dfrac{NL}{AE}\right] + \left[\sum \displaystyle\int_0^L \left(\frac{\partial M}{\partial \bar{M}}\right)\frac{M}{EI} dx\right] + \left[\displaystyle\int_0^L \left(\frac{\partial V}{\partial \bar{M}}\right)\frac{V}{GA_v} dx\right] \end{cases}$$

| Beam, Loading, and Bending Moment Diagram | Equations for Slope and Deflection |
|---|---|
| <br> | $0 \le x \le \frac{L}{2}:$<br>$\theta = \frac{P}{16EI}(4x^2 - L^2)$<br>$y = \frac{P}{48EI}(4x^3 - 3L^2x)$<br>$\theta_A = -\frac{PL^2}{16EI};\ \ \theta_B = \frac{PL^2}{16EI}$<br>$y_{\max} = -\frac{PL^3}{48EI}$ |
| <br> | $0 \le x \le a:$<br>$\theta = \frac{Pb}{6EIL}(3x^2 + b^2 - L^2)$<br>$y = \frac{Pb}{6EIL}(x^3 + b^2x - L^2x)$<br><br>$a \le x \le L:$<br>$\theta = \frac{Pa}{6EIL}[L^2 - a^2 - 3(L-x)^2]$<br>$y = \frac{Pa(L-x)}{6EIL}(x^2 + a^2 - 2Lx)$<br>$\theta_A = -\frac{Pb}{6EIL}(L^2 - b^2)$<br>$\theta_B = \frac{Pa}{6EIL}(L^2 - a^2)$<br><br>For $a \ge b:$<br>$y_{\max} = -\frac{Pb}{9\sqrt{3}EIL}(L^2 - b^2)^{3/2}$<br>at $x = \left(\frac{L^2 - b^2}{3}\right)^{1/2}$ |
| <br> | $\theta = -\frac{M}{6EIL}(3x^2 - 6Lx + 2L^2)$<br>$y = -\frac{M}{6EIL}(x^3 - 3Lx^2 + 2L^2x)$<br>$\theta_A = -\frac{ML}{3EI};\ \ \theta_B = \frac{ML}{6EI}$<br>$y_{\max} = -\frac{ML^2}{9\sqrt{3}EI}$<br>at $x = L\left(1 - \frac{1}{\sqrt{3}}\right)$ |

| Beam, Loading, and Bending Moment Diagram | Equations for Slope and Deflection |
|---|---|
|  | $0 \le x \le a:$<br>$\theta = \frac{M}{6EIL}(-3x^2 + 6aL - 3a^2 - 2L^2)$<br>$y = \frac{M}{6EIL}(-x^3 + 6aLx - 3a^2x - 2L^2x)$<br>$\theta_A = \frac{M}{6EIL}(6aL - 3a^2 - 2L^2)$<br>$\theta_B = \frac{M}{6EIL}(L^2 - 3a^2)$ |
|  | $\theta = -\frac{w}{24EI}(4x^3 - 6Lx^2 + L^3)$<br>$y = -\frac{w}{24EI}(x^4 - 2Lx^3 + L^3x)$<br>$\theta_A = -\frac{wL^3}{24EI}$<br>$\theta_B = \frac{wL^3}{24EI}$<br>$y_{max} = -\frac{5wL^4}{384EI}$ at $x = \frac{L}{2}$ |
|  | $0 \le x \le a:$<br>$\theta = -\frac{w}{24EIL}[4Lx^3 - 6a(2L-a)x^2 + a^2(2L-a)^2]$<br>$y = -\frac{w}{24EIL}[Lx^4 - 2a(2L-a)x^3 + a^2(2L-a)^2x]$<br>$a \le x \le L:$<br>$\theta = -\frac{wa^2}{24EIL}(6x^2 - 12Lx + a^2 + 4L^2)$<br>$y = -\frac{wa^2}{24EIL}(L-x)(-2x^2 + 4Lx - a^2)$<br>$\theta_A = -\frac{wa^2}{24EIL}(2L-a)^2$<br>$\theta_B = \frac{wa^2}{24EIL}(2L^2 - a^2)$ |
|  | $\theta = -\frac{w}{360EIL}(15x^4 - 30L^2x^2 + 7L^4)$<br>$y = -\frac{w}{360EIL}(3x^5 - 10L^2x^3 + 7L^4x)$<br>$\theta_A = -\frac{7wL^3}{360EI}$<br>$\theta_B = \frac{wL^3}{45EI}$<br>$y_{max} = -0.00652\frac{wL^4}{EI}$ at $x = 0.5193L$ |

# BENDING MOMENTS, SLOPES, AND DEFLECTIONS OF BEAMS UNDER VARIOUS LOADING CONDITIONS




| Beam, Loading, and Bending Moment Diagram | Equations for Slope and Deflection |
| --- | --- |
|   | $0 \le x \le a:$<br>$\theta = \frac{P}{2EI}(x^2 - 2ax)$<br>$y = \frac{P}{6EI}(x^3 - 3ax^2)$<br>$a \le x \le L:$<br>$\theta = -\frac{Pa^2}{2EI}$<br>$y = \frac{Pa^2}{6EI}(a - 3x)$<br>$\theta_B = -\frac{Pa^2}{2EI};\ y_B = -\frac{Pa^2}{6EI}(3L - a)$ |
|   | $0 \le x \le a:$<br>$\theta = -\frac{Mx}{EI}$<br>$y = -\frac{Mx^2}{2EI}$<br>$a \le x \le L:$<br>$\theta = -\frac{Ma}{EI}$<br>$y = \frac{Ma}{2EI}(a - 2x)$<br>$\theta_B = -\frac{Ma}{EI};\ y_B = -\frac{Ma}{2EI}(2L - a)$ |
|   | $0 \le x \le a:$<br>$\theta = \frac{w}{6EI}(3ax^2 - 3a^2x - x^3)$<br>$y = \frac{w}{24EI}(4ax^3 - 6a^2x^2 - x^4)$<br>$a \le x \le L:$<br>$\theta = -\frac{wa^3}{6EI}$<br>$y = \frac{wa^3}{24EI}(a - 4x)$<br>$\theta_B = -\frac{wa^3}{6EI};\ y_B = -\frac{wa^3}{24EI}(4L - a)$ |
|   | $0 \le x \le a:$<br>$\theta = \frac{w}{24EIa}(x^4 - 4ax^3 + 6a^2x^2 - 4a^3x)$<br>$y = \frac{w}{120EIa}(x^5 - 5ax^4 + 10a^2x^3 - 10a^3x^2)$<br>$a \le x \le L:$<br>$\theta = -\frac{wa^3}{24EI}$<br>$y = \frac{wa^3}{120EI}(-5x + a)$<br>$\theta_B = -\frac{wa^3}{24EI};\ y_B = -\frac{wa^3}{120EI}(5L - a)$ |