1. Suppose government expenditures (GOV) contribute to production because government spending makes private capital (K) and labor (L) more productive. Write the production function as:

$$Y_t = K_t^{\alpha} GOV_t^{\gamma} (L_t A_t)^{\beta} \quad \text{where} \quad \alpha + \beta + \gamma = 1$$

and A represents productivity. Assume for simplicity that the government runs a balanced budget where taxes (T) are always equal to government expenditures. Consumers save (S) a fixed fraction (φ) of disposable income:

$$S_t = \phi(Y_t - T_t).$$

Capital evolves according to the following equation where δ is a constant depreciation rate:

$$I_t = \frac{dK_t}{dt} + \delta K_t.$$

This is a closed economy where savings always is equal to investment. Let n be the growth rate of labor supply and let g be the growth rate of productivity where g and n are constant parameters. Assume all parameters are positive.

A. Re-write each equation in per-units of effective labor, using lower-case letters (y, k, i and s) to describe quantities in per-unit of effective labor, except for, $G_t = \frac{GOV_t}{A_t L_t}$)

B. Derive the steady-state condition for k in terms of parameters and the level of government spending per-unit of effective labor.

C. What is the level of G that maximizes steady state y? Write your solution for steady state G as a linear function of steady state k with constant coefficients.

D. What is the relevant sufficient second order condition for this problem in part C? (You don't have to prove that this condition holds - it is fairly tedious and may require cerrtain assumptions about parameter values)

2. Consider a model in which consumers live for two periods, work (n) in the first period of life and retire in the second period. Let utility be given by

$$\log(c_1) + b[\log(1-n)] + \frac{\log(c_2)}{1+\rho}$$

where n can be thought of as the share or time spent working (so 1-n is the share of time taking leisure) while b and ρ are positive constant parameters. Consumers are subject to two budget constraints:

$$s + c_1 = wn \quad \text{and} \quad c_2 = s\,(1+r)$$

where consumers choose consumption (c_1), savings (s) and labor effort (n) when young and consumption when old (c_2). Real wages and interest rates are taken as given by the consumer.

A. Set up the Lagrangian for the consumer to maximize utility subject to these resource constraints.

B. Derive first order conditions for this optimization problem.

C. Derive the effects of each exogenous variable (r, w) on each of the endogenous variables (c_1, s, n, and c_2). For each of these 8 effects, state whether it is positive, negative, zero, or ambiguous.