

University of Kansas, Department of Economics  
Economics 911: Applied Macroeconomics

Problem Set 2: Multivariate Time Series Analysis

Unless stated otherwise, assume that shocks (e.g.  $\varepsilon$  and  $\mu$ ) are white noise in the following questions.

1. Consider the following multivariate process.

$$y_t = y_{t-1} + \varepsilon_t + \theta\varepsilon_{t-1}$$

$$x_t = \phi y_{t-1} + \mu_t$$

- i) Is this vector process stationary?  
ii) Is the vector process of first differences ( $\Delta y, \Delta x$ ) invertible?

2. Is the following process invertible?

$$y_{1,t} = \varepsilon_{1,t} + .5\varepsilon_{1,t-1} - .3\varepsilon_{2,t-1}$$

$$y_{2,t} = \varepsilon_{2,t} + .6\varepsilon_{2,t-1} + .7\varepsilon_{1,t-1}$$

3. Suppose that the process  $(y,x)$  has the following vector autoregressive representation.

$$y_t = .4y_{t-1} + .4x_{t-1} + \varepsilon_t$$

$$x_t = .4y_{t-1} + .4x_{t-1} + \mu_t$$

- i) Is this multivariate process stationary?  
ii) Calculate the univariate representations for  $y$  and  $x$ , respectively.

4. Prove that if a multivariate time series process has autoregressive roots on the unit circle, then the corresponding univariate processes will also have autoregressive roots on the unit circle.

5. Suppose the RBC modelers are correct and that exogenous shocks to the money supply have no effect on output. Assume that output is a function of one lag of an exogenous variable that econometricians do not observe and an independent white noise shock. Also assume that the central bank adjusts money to the current value of this exogenous variable that econometricians don't observe and a different white noise shock that is independent. Will money fail to Granger cause output in the bivariate VAR?

6. Suppose that  $y$  is an  $n$ -vector of time series that is second difference stationary and has the following

Wold representation:

$$\Delta^2 y_t = \alpha + C(L)\varepsilon_t \text{ with } \alpha \text{ an } n \times 1 \text{ vector, } C(L) = \sum_{i=0}^{\infty} C_i L^i$$

and each  $C_i$  an  $n \times n$  matrix. Suppose that the presample values of  $\varepsilon$  are equal to zero. Assuming initial conditions  $y_0, y_{-1}, y_{-2}, \dots$  for the vector of time series, derive the process for  $y$  (in levels)? Are the deterministic and stochastic elements for this process fundamentally different from the case where  $y$  is stationary after first differencing? If so, please explain. If there are some linear combinations of  $\Delta y$  that are cointegrated, what kind of restrictions would that impose on  $C(L)$ ? If there are some linear combinations of  $y$  that are cointegrated, what kind of restrictions would that impose on  $C(L)$ ?

7. Consider the following bivariate VAR model:

$$\Delta y_t = .3\Delta y_{t-1} + e_t^y$$

$$\Delta m_t = .9\Delta m_{t-1} + e_t^m$$

$$\text{where: } e_t = (e_t^y, e_t^m)' \text{ and } Ee_t e_t' = \begin{bmatrix} 1.0 & .5 \\ .5 & 2.0 \end{bmatrix}$$

and  $e_t$  is not serially correlated.

Derive an alternative representation which has shocks that are uncorrelated and the matrix of long-run multipliers for these shocks is lower triangular. Derive the impulse response function and variance decomposition for  $\Delta y$ ,  $\Delta m$ ,  $y$  and  $m$  from this representation.

8. Consider the following bivariate VAR model.

$$x_t = .7x_{t-1} + e_t^x$$

$$y_t = .5y_{t-1} + e_t^y$$

$$\text{where: } e_t = (e_t^x, e_t^y)' \text{ and } Ee_t e_t' = \begin{bmatrix} 1.0 & .5 \\ .5 & 2.0 \end{bmatrix}$$

This model can be viewed as a special case of a general model given as the vector  $Z_t = (x_t, y_t)'$  and

$$\Phi_0 Z_t = \Phi_1 Z_{t-1} + e_t \text{ with } \Phi_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \Phi_1 = \begin{bmatrix} .7 & 0 \\ 0 & .5 \end{bmatrix} \text{ in this particular model.}$$

(a) Derive an alternative representation of the data with uncorrelated disturbances and a lower triangular  $\Phi_0$  matrix. This representation implies that  $x_t$  depends on lagged values of the data plus a white noise innovation, while  $y_t$  depends on current  $x_t$ , lagged values of the data plus a white noise innovation.

(b) Derive the IRF and VDF of  $x_t$  and  $y_t$  with respect to the innovations in the representation in

part (a).

9. Consider the bivariate VAR model:  $\beta(L)Z_t = e_t$  where  $Z_t = (x_t, y_t)'$ ,

$$\beta_{ii}(L) = 1 - a_{ii}L - b_{ii}L^2 \text{ for } i=1,2, \quad \beta_{12} = -a_{12}L - b_{12}L^2, \quad \beta_{21} = -a_{21}L - b_{21}L^2$$

$$e_t = (e_{xt}, e_{yt})', \quad Ee_{vt}e_{wt} = \sigma_{vw} \text{ for } v = x \text{ or } y \text{ and } w = x \text{ or } y, \quad Ee_{vt}e_{w\tau} = 0 \text{ when } t \neq \tau, \text{ and}$$

$\beta(L)$  is invertible.

- What conditions must hold on  $\beta(L)$  parameters, for a finite VAR in first differences to exist?
- What condition must hold on  $\beta(L)$  parameters for there to exist a Vector Error Corrections Model with only one cointegrating vector?
- What condition must hold on  $\beta(L)$  parameters if the two variables in the model are stationary.

For the remaining parts of this question, assume the series are stationary in levels.

- Parameterize the univariate representation for each series (assuming there are no common factors.)
- Construct the non-orthogonalized moving average representation associated with each innovation.
- Construct the moving average representation for each shock obtained using a Cholesky ordering with  $x$  placed first.

10. Consider the bivariate VAR model in first differences

$$\beta(L)\Delta Z_t = e_t \text{ where}$$

$$\Delta Z_t = (\Delta y_t, \Delta m_t)' \text{ and } e_t = (e_{yt}, e_{mt})'$$

Assume that  $Ee_{vt}e_{w\tau} = \sigma_{vw}$  if  $t=\tau$   
 $= 0$  otherwise.

Let  $\beta(L) = I - \beta_1 L - \beta_2 L^2$  where  $\beta_1 = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$ ,  $\beta_2 = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix}$  and the inverse of

$B(L)$  exists.

- Can an additional condition be imposed on  $\beta(L)$  to yield a Vector Error Corrections Model with stationary linear combinations of  $Z_t$  under these assumptions? If so, what is that condition?

Ignore your answer to part (a) in the remainder of this question.

- Assuming no common factors, construct the univariate representation for output ( $y$ ) and money ( $m$ ). What ARIMA process does each series follow?

c) Construct the moving average representation for  $Z_t$  assuming exogenous shocks to money are uncorrelated with exogenous shocks to output (therefore it is convenient to normalize the covariance matrix for exogenous shocks to be an identity) and exogenous shocks to money have no long-run effect on the level of output. Write the MAR as a function of the VAR parameters and the variances and covariances of the VAR innovations. (You need to show precisely how to calculate the MAR, but you need not multiply out each matrix.)

11. Suppose that the rational expectations theory of the term structure of interest rates is correct and the 2-period interest rate ( $R$ ) is an average of the current and expected future one-period interest rate ( $r$ ) and a risk premium ( $\gamma$ ):

$$R_t = \left(\frac{1}{2}\right)(r_t + E_t r_{t+1}) + \gamma_t,$$

where the risk premium is dynamic and a function of  $r_{t-1}$ ,  $R_{t-1}$  and  $\varepsilon_t$  which is the serially uncorrelated risk premium shock which has constant standard deviation  $\sigma_\varepsilon$ :

$$\gamma_t = \alpha_1 r_{t-1} + \alpha_2 R_{t-1} + \varepsilon_t.$$

$E_t$  indicates expectations based on all information known at time  $t$ . Suppose the reduced form for  $r$  is a function of one lag of each variable:

$$r_t = \beta_1 r_{t-1} + \beta_2 R_{t-1} + e_{rt}.$$

A. Write an equation for  $R_t$  as a function of  $r_t$ ,  $r_{t-1}$ ,  $R_{t-1}$ , and  $\varepsilon_t$  showing exactly how the coefficients from the reduced equation for  $r$  and the economic structure determine the parameters in this equation.

B. Your result in part A can be rewritten as:

$$R_t = \tau_0 r_t + \tau_1 r_{t-1} + \tau_2 R_{t-1} + \varepsilon_t^*,$$

where  $\varepsilon^*$  is a linear function of  $\varepsilon$  and each  $\tau$  is a constant coefficient. Suppose the short rate is given by the following central bank policy reaction function:

$$r_t = \pi_0 R_t + \pi_1 r_{t-1} + \pi_2 R_{t-1} + \mu_t,$$

where the central bank responds to current values of the 2-period rate and lagged values of 1 and 2 period interest rates and a monetary policy shock ( $\mu$ ). Prove that this structural system can also be written as the following function of residuals from a bivariate VAR:

$$e_{Rt} = \tau_0 e_{rt} + \varepsilon_t^* \tag{1}$$

$$e_{rt} = \pi_0 e_{Rt} + \mu_t. \tag{2}$$

C. Why can't OLS be used to estimate the second equation? Given that you know  $\tau_0$  from the term structure theory and that the variance covariance matrix for the VAR's residuals is given by:

$$E \begin{bmatrix} e_t^R \\ e_t^r \end{bmatrix} \begin{bmatrix} e_t^R & e_t^r \end{bmatrix} = \begin{bmatrix} \sigma_R^2 & \sigma_{rR} \\ \sigma_{rR} & \sigma_r^2 \end{bmatrix},$$

how would you estimate  $\pi_0$ ? What would this estimate converge to asymptotically (i.e. in large sample)? [Hint: While you can't use OLS to obtain a consistent estimate of

$\pi_0$ , if you did estimate equation 2 by OLS,  $\pi_0$  would converge asymptotically to  $\frac{E e_{rt} e_{Rt}}{E e_{Rt}^2} = \frac{\sigma_{rR}}{\sigma_R^2}$ .]

12. Many economists have found evidence that each interest rate has a unit root and that different interest rates are cointegrated. (While the theory of the term structure of interest rates can be used to justify this finding, you can ignore that theory in answering this question.) Consider the example from the previous question. If each rate has a unit root and the interest rate spread,  $R-r$ , is stationary, the following VECM can be estimated:

$$\begin{aligned} b_{11}(L)\Delta R_t + b_{12}(L)\Delta r_t &= \alpha_1(R_{t-1} - r_{t-1}) + e_t^R \\ b_{21}(L)\Delta R_t + b_{22}(L)\Delta r_t &= \alpha_2(R_{t-1} - r_{t-1}) + e_t^r \end{aligned}$$

where each  $b_{ij}(L)$  is a lag polynomial of a common length for all combinations of  $i$  and  $j$ ,  $b_{ii}(0)=1$  for all  $i$  (which means the first equation pertains to  $R_t$  and the second to  $r_t$ ),  $b_{ij}(0)=0$  for  $i$  not equal to  $j$ ,  $\alpha_1$  and  $\alpha_2$  are coefficients and the last term in each equation is the residual.

A. Show how the VECM can be rewritten as a VAR in the levels of  $R$  and  $r$ . Are this VAR model residuals the same residuals as the residuals in VECM?

B. Show how the VECM can be rewritten as a VAR in  $\Delta r$  and  $(R-r)$ . Show whether or not these residuals are the same as the residuals in the VECM?

C. Using the structure in equations 1 and 2 from the previous question, derive impulse responses of  $\Delta R$  and  $\Delta r$  to the structural shocks ( $\varepsilon^*$  and  $\mu$ ). Report the lag polynomials associated with each of these responses.

13. Inflation ( $\pi$ ) output ( $y$ ) and the interest rate ( $R$ ) are variables in the following three reduced-form equations:

$$\begin{aligned} \pi_t &= \beta_{11}\pi_{t-1} + \beta_{12}y_{t-1} + \beta_{13}R_{t-1} + e_t^\pi \\ y_t &= \beta_{21}\pi_{t-1} + \beta_{22}y_{t-1} + \beta_{23}R_{t-1} + e_t^y \\ R_t &= \beta_{31}\pi_{t-1} + \beta_{32}y_{t-1} + \beta_{33}R_{t-1} + e_t^R \end{aligned}$$

The variances and covariances of the residuals are given by:

$$Ee_t^i e_t^j = \sigma_{ij}, \quad Ee_t^i e_\tau^j = 0 \text{ for } t \neq \tau, \quad \text{with } i=\pi,y,R \text{ and } j=\pi,y,R.$$

For parts A, B and C, you may use lag polynomials to calculate IRFs:

- A) Calculate IRFs for the  $(\pi,y,R)$  Cholesky ordering
- B) Show that the IRFs for the first shock in the  $(\pi,R,y)$  Cholesky ordering are precisely the same as IRFs for the first shock in part A.
  - ii) Show that the IRFs for the third shock in the  $(y,\pi,R)$  Cholesky ordering are precisely the same as IRFs for the third shock in part A.

For parts D, E and F, assume  $\beta_{11} = \beta_{21} = \beta_{31} = 0$ .

D. Can OLS be used to obtain consistent and efficient estimates of the  $\beta_{12}$ ,  $\beta_{22}$ ,  $\beta_{32}$ ,  $\beta_{13}$ ,  $\beta_{23}$  and  $\beta_{33}$  parameters? Explain your reasoning.

E. Assume that monetary policy shocks ( $\varepsilon_t^{ms}$ ) immediately affect R but have no immediate effect on y or  $\pi$ . Also assume that monetary policy shocks are independent of the structural shocks to inflation and output (call them  $\varepsilon_t^y$  and  $\varepsilon_t^\pi$ ) at all points in time. Assume these other two structural shocks have immediate and simultaneous effects on inflation and output. Using lag polynomials, calculate the impulse response function of each variable to a monetary policy shock. In this calculation, normalize the structural shocks to have unit variance.

F. Suppose  $\beta_{32}=0$ . What exactly is the impulse response of y to a shock to money supply at an arbitrary point in time ( $\frac{dy_{t+h}}{d\varepsilon_t^{ms}}$ )? Write this answer in terms of parameters, not in terms of the lag polynomials.

14. In contrast to the previous question, suppose that the economic structure is given by

$$\begin{aligned}\pi_t &= \alpha_1 E_t \pi_{t+1} + \alpha_2 y_t + \alpha_3 \pi_{t-1} + \alpha_4 y_{t-1} + \alpha_5 R_{t-1} + \varepsilon_t^{as} \\ y_t &= \gamma_1 E_t y_{t+1} + \gamma_2 R_t + \gamma_3 \pi_{t-1} + \gamma_4 y_{t-1} + \gamma_5 R_{t-1} + \varepsilon_t^{ad} \\ R_t &= \phi_1 \pi_t + \phi_2 y_t + \phi_3 \pi_{t-1} + \phi_4 y_{t-1} + \phi_5 R_{t-1} + \varepsilon_t^{ms}\end{aligned}$$

where  $E_t$  denotes rational expectations based on time t information and

$$\begin{aligned}E\varepsilon_t^i \varepsilon_t^j &= 0 \quad \text{if } i \neq j \\ &= \sigma_i^2 \quad \text{if } i = j\end{aligned}$$

for  $i=as,ad,ms$  and  $j=as,ad,ms$

$$E\varepsilon_t^i \varepsilon_\tau^j = 0 \quad \text{if } t \neq \tau \text{ (for all } i \text{ and } j).$$

A. Carefully explain why this model does not impose any over-identifying (i.e. testable) restrictions on the reduced form VAR, which is given by the unconstrained version of the first 3 equations in the previous question.

B. Suppose you know the values for parameters  $\phi_1$ ,  $\phi_2$  and  $\gamma_1$ . How would you estimate all the other parameters in the 3 structural equations? Be specific about methods and moment conditions.

C. Show precisely what the parameters  $\alpha_1$ ,  $\alpha_2$  and  $\gamma_2$  are equal to in terms of moments. (But don't grind out the relationship between each moment and the covariance matrix of residuals.)

15. Assume x and y are each integrated of order 1 and that the structure is given by:

$$\begin{aligned}\Delta x_t &= \Theta_{11}(L)\varepsilon_t^1 + \Theta_{12}(L)\varepsilon_t^2 \\ \Delta y_t &= \Theta_{21}(L)\varepsilon_t^1 + \Theta_{22}(L)\varepsilon_t^2\end{aligned}$$

where each structural shock is uncorrelated with all other structural shocks (at all lags and leads) and each structural shock has unit variance. Assume you have estimated a VAR for this bivariate system and obtained a variance covariance matrix for residuals,  $\Sigma_e$ , and a matrix associated with the sum of coefficients in the VAR,  $\beta(1)$ . (Technically, the  $\beta(1)$  matrix is equal to the identity matrix less the sum of VAR coefficients matrix). Let the matrix of long-run multipliers be given by

$$\begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}.$$

Assume that x and y are NOT cointegrated in parts A and B.

A. For convenience define

$$M = \beta(1)^{-1} \Sigma_e [\beta(1)']^{-1} \quad \text{where} \quad M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{bmatrix}.$$

Derive the values of the other 3 long-run multipliers for any given value of  $\rho_{21}$ .

B. Now assume that  $\rho_{21}=0$  and that you have correctly identified all the  $\theta(L)$  parameters. Derive the fraction of forecast error variance for  $y_{t+h}$  explained by shock  $\varepsilon_t^1$ . Show and explain what this variance will converge to as h goes to  $\infty$ . (Important reminder: This variance pertains to the level, not the first difference of y.)

C. Now assume x and y are cointegrated. Let  $y_t - \lambda x_t$  be the stationary linear combination (i.e. the cointegrating vector) for any non-zero value of  $\lambda$ . Specifically what non-linear restriction does cointegration impose on the long-run multipliers?

16. Bullard and Keating (JME 1995) estimate bivariate VAR models of the change in inflation ( $\Delta\pi$ ) and the change in the log of output ( $\Delta y$ ). Using a variation on the permanent and transitory shock decomposition for output that Blanchard and Quah used to identify aggregate supply and demand effects, Bullard and Keating identify permanent and transitory shocks to inflation (note that both types of shocks may have a permanent effect on the level of output in this framework). Specifically, they develop the following statistical model:

$$\begin{aligned} \Delta \pi_t &= R_{11}(L)v_t^1 + R_{12}(L)v_t^2 \\ \Delta y_t &= R_{21}(L)v_t^1 + R_{22}(L)v_t^2 \end{aligned}$$

which is identified by setting  $\sum_{k=0}^{\infty} R_{12k} = R_{12}(1) = 0$  and by restricting  $E v_t^1 v_t^2 = 0$ . For

convenience, assume each shock in this statistical model has unit variance.

One criticism of this approach is that permanent changes in money growth are not exogenous, but instead obtain because the money supply is endogenous to real activity. It is argued that increases in inflation were primarily the result of a central bank increasing money growth in response to recessions that were caused by adverse supply shocks. In other words, critics would model the structure as:

$$\Delta \pi_t = \theta_{11}(L)\varepsilon_t^{\Delta m} + \theta_{12}(L)\varepsilon_t^{\text{as}} \quad (3)$$

$$\Delta y_t = \theta_{21}(L)\varepsilon_t^{\Delta m} + \theta_{22}(L)\varepsilon_t^{\text{as}} \quad (4)$$

where  $\varepsilon^{\text{as}}$  is an exogenous aggregate supply shock,  $\varepsilon^{\Delta m}$  is an exogenous money growth shock with

$$\sum_{k=0}^{\infty} \theta_{12k} = \theta_{12}(1) < 0 \text{ because adverse supply shocks cause the Fed to permanently raise money}$$

growth in an attempt to partially offset the decline in output. For convenience, assume the two structural shocks are uncorrelated and each shock has variance equal to one (just as was done with the statistical model).

[Hint: Consider how the structure and the statistical model will map into the reduced form.]

A. Show precisely how  $R_{21}(1)$  is biased from  $\theta_{21}(1)$ , assuming  $\theta_{11}(1) > 0$  and  $\theta_{22}(1) > 0$ . What is the direction of the bias (upward, downward, toward zero, away from zero, ambiguous)?

B. Bullard and Keating find for most countries that permanent shocks to inflation (which virtually every theory tells us are associated with permanent changes in money growth) are NOT associated with permanent changes in the level of output. The primary exception is most of the low inflation countries for which they find that a permanent increase in inflation leads to a permanent INCREASE in the level of output. The critics argue that exogenous money growth shocks (i.e. exogenous inflation shocks) have no long-run output effect, and that Bullard and Keating get their empirical result because of the endogenous response of money growth to aggregate supply shocks. Based on your analysis in part A, are the critics right or wrong? Why?

C. Modify the theoretical framework in equations 3 and 4 by assuming  $\sum_{k=0}^{\infty} \theta_{12k} = \theta_{12}(1) = 0$

and  $\theta_{12}(0) = 0$ . The idea behind these restrictions is that the Fed doesn't allow the inflation rate in the short run or the long run to react to aggregate supply shocks (ignoring the implausibility for these restrictions). Would these theoretical restrictions impose any testable restrictions on the reduced form's coefficients or on the covariance matrix for its residuals? If so what exactly would these restrictions be?

17. Suppose the economy can be described by the following VARMA model:

$$\Phi(L)y_t = \Theta(L)e_t$$

where  $y_t = (x_t, q_t)'$ ,  $\Phi_{ii}(L) = 1 - a_{ii}L$  and  $\Theta_{ii}(L) = 1 + b_{ii}L$  for  $i=1,2$ ,

$\Phi_{12}(L) = -a_{12}L$ ,  $\Phi_{21}(L) = -a_{21}L$ ,  $\Theta_{12}(L) = b_{12}L$ ,  $\Theta_{21}(L) = b_{21}L$ ,

Also,  $e_t = (e_{xt}, e_{qt})'$ ,  $Ee_{vt}e_{wt} = \sigma_{vw}$  for  $v = x$  or  $q$  and  $w = x$  or  $q$ , and  $Ee_{vt}e_{w\tau} = 0$  for all  $t \neq \tau$ .

A. What condition(s) must hold on the parameters for the vector process to be stationary?



B. What condition(s) must hold on the parameters for the vector process to be invertible?

C. Assuming the vector process is stationary and invertible, derive the univariate representation for  $x_t$ . Assuming no common factors in the lag polynomials, what type of ARMA process is it?

D. Assuming the vector process is stationary and invertible, construct the impulse response for each variable to the second shock from a short-run recursive model in which  $x$  is placed first in the ordering.

E. What condition(s) must hold on the parameters of the VARMA if both variables are difference stationary and there is no cointegration?

18. In each of the following structural equations no restrictions are placed on dynamics. In addition to unconstrained dynamics:

Equation (i) uses the rational expectations theory of the term structure of interest rates to relate the 2-period interest rate ( $R$ ) to an average of the current and expected future one-period interest rate ( $r$ ) as well as a shock to the term structure;

Equation (ii) is an Old-Keynesian IS curve with output as a function of the short-term real rate of interest and an IS shock;

Equation (iii) is a Taylor rule that specifies the short-term nominal rate as a function of inflation, output and a monetary policy shock; and

Equation (iv) writes inflation as a function of short and long term interest rates, output and an aggregate supply shock:

$$\begin{aligned}
 \text{(i)} \quad R_t &= \left(\frac{1}{2}\right)(r_t + E_t r_{t+1}) + \phi'_R Z_{t-1} + \varepsilon_t^{\text{TS}} \\
 \text{(ii)} \quad y_t &= \gamma[r_t - E_t \pi_{t+1}] + \phi'_y Z_{t-1} + \varepsilon_t^{\text{IS}} \\
 \text{(iii)} \quad r_t &= \theta_1 \pi_t + \theta_2 y_t + \phi'_r Z_{t-1} + \varepsilon_t^{\text{MP}} \\
 \text{(iv)} \quad \pi_t &= \alpha_1 R_t + \alpha_2 y_t + \alpha_3 r_t + \phi'_\pi Z_{t-1} + \varepsilon_t^{\text{AS}}
 \end{aligned}$$

where  $E_t$  denotes rational expectations based on time  $t$  information. Let  $Z_{t-1}$  be the set of  $k$  lagged

endogenous variables. In other words, we can write  $X_{t-j} = \begin{bmatrix} R_{t-j} \\ \pi_{t-j} \\ y_{t-j} \\ r_{t-j} \end{bmatrix}$  for  $j=0,1,2,..k$  and then

$$Z_{t-1} = \begin{bmatrix} X_{t-1} \\ \vdots \\ X_{t-k} \end{bmatrix}. \text{ Each } \phi_j' \text{ is a vector of coefficients for structural equations and } j=R,\pi,y,r.$$

For all  $i,j$  from the set  $\{TS,IS,MP,AS\}$ :

$$\begin{aligned} E\varepsilon_t^i \varepsilon_t^j &= 0 & \text{if } i \neq j \\ &= \sigma_i^2 & \text{if } i = j \end{aligned} \quad \text{for } i \text{ and } j = TS, IS, MP, AP$$

$$E\varepsilon_t^i \varepsilon_\tau^j = 0 \quad \text{if } t \neq \tau \text{ (for any } i \text{ and } j).$$

A. Re-write each structural equation in terms of the relationship between VAR innovations, structural parameters and structural shocks. Assume the VAR is written as:

$$\beta(L)X_t = e_t \text{ where } \beta(L) = I - \beta_1 L - \beta_2 L^2 - \dots - \beta_k L^k, \quad \beta_i = \begin{bmatrix} \beta_i^{RR} & \beta_i^{R\pi} & \beta_i^{Ry} & \beta_i^{Rr} \\ \beta_i^{\pi R} & \beta_i^{\pi\pi} & \beta_i^{\pi y} & \beta_i^{\pi r} \\ \beta_i^{yR} & \beta_i^{y\pi} & \beta_i^{yy} & \beta_i^{yr} \\ \beta_i^{rR} & \beta_i^{r\pi} & \beta_i^{ry} & \beta_i^{rr} \end{bmatrix}$$

$$\text{for } i=1,2,\dots,k \text{ and } e_t = (e_t^R, e_t^\pi, e_t^y, e_t^r)'$$

B. How would you estimate the parameters  $\gamma, \theta_1, \theta_2, \alpha_1, \alpha_2$  and  $\alpha_3$ ? For each equation, be specific about the left hand side variable, the regressor(s) and, if necessary, any variables that you are using as instruments.

C. Suppose you are NOT willing to impose restrictions from equations (ii), (iii) and (iv). However, you would still like to identify the impulse response function of each variable to the term premium shock. It turns out that this can be done using a particular Cholesky decomposition! Prove it.

One easier way to show this is first by rewriting the structural equations in terms of residuals, structural parameters and structural shocks as:  $\begin{bmatrix} \bar{\delta}_{11} & \bar{\delta}_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix} \begin{bmatrix} e_t^R \\ e_t^V \end{bmatrix} = \begin{bmatrix} \varepsilon_t^{TS} \\ \varepsilon_t^V \end{bmatrix}$  where

$e_t^V = (e_t^\pi, e_t^y, e_t^r)'$  and  $\varepsilon_t^V = (\varepsilon_t^{AS}, \varepsilon_t^{IS}, \varepsilon_t^{MP})'$ . I've placed bars over the  $\delta$  coefficients that are known because of the term structure theory. The other  $\delta$  coefficients are unknown because we now are assuming that we don't want to impose restrictions from any of the other equations. Re-write this system of equations for residuals such that a Cholesky decomposition can be used to

identify the effects of the term structure shock.

Show what matrix you would perform a Cholesky decomposition on and how it would be used to identify the response of each variable to the term structure shock. For convenience,

you can write the covariance matrix for residuals as:  $\begin{bmatrix} \Sigma_{RR} & \Sigma_{RV} \\ \Sigma_{VR} & \Sigma_{VV} \end{bmatrix}$  where, for example,  $\Sigma_{RR}$  is the variance of the residuals to the equation for R from the VAR.

19. Suppose the following triangular representation --- which is assumed to be structural, not a reduced form --- characterizes the relationship between variables x and y:

$$\begin{bmatrix} \Delta y_t \\ s_t \end{bmatrix} = \begin{bmatrix} \theta_{11}(L) & \theta_{12}(L) \\ \theta_{21}(L) & \theta_{22}(L) \end{bmatrix} \begin{bmatrix} \tau_t \\ \eta_t \end{bmatrix}$$

where each element in  $\tau$  and in  $\eta$  is an independent white noise shock with variance normalized to 1,  $\tau$  are the structural shocks that have permanent effects on at least some of the variables,  $\eta$  are the structural shocks that have no permanent effects on any variables and  $s_t = x_t - \delta y_t$  is one way to represent all of the stationary linear combinations of variables. Assume that  $\theta(1)$  is full rank and that  $\theta(L)$  is invertible. We know that when  $\eta$  shocks have temporary effects,  $\theta_{12}(1) = 0$ , however,  $\theta_{21}(1)$  and  $\theta_{22}(1)$  are unrestricted in this type of model. Assume there are  $n_x (> 1)$  variables in x,  $n_y (> 1)$  variables in y, and therefore  $\delta$  is an  $n_x \times n_y$  matrix of parameters.

A. Given a VAR that is written as:  $\beta(L)X_t = e_t$  where  $X_t = [\Delta y_t', s_t']'$  and a covariance matrix for residuals given by  $\Sigma_e$ , show and explain precisely how you could estimate the matrix of parameters  $\theta_{11}(1)$  if this matrix is lower triangular (i.e. the long-run relationship is recursive). In your answer, you might find it convenient to let  $M = \beta(1)^{-1} \Sigma_e \beta(1)^{-1}$  and to

write  $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$  with the dimensions of  $M_{ij}$  conformable to  $\theta_{ij}$ . (If you get stuck on this question, Keating (2002, Macro Dynamics) may be a good place to look for inspiration).

B. Show that the impulse responses of  $\Delta y$  and  $s$  to the  $\tau$  shocks are identified, using matrices from the VAR and the model from part A.

C. The structural triangular representation given at the beginning of this question, can

also be written as follows:  $\begin{bmatrix} A_{11}(L) & A_{12}(L) \\ A_{21}(L) & A_{22}(L) \end{bmatrix} \begin{bmatrix} \Delta y_t \\ s_t \end{bmatrix} = \begin{bmatrix} \tau_t \\ \eta_t \end{bmatrix}$  where the  $A(L)$  matrix is simply equal to the inverse of  $\theta(L)$ . Using this version of the triangular representation, derive the

structural vector error correction model with precisely the same structural errors. [Hint: Consider the multivariate version of the Beveridge-Nelson decomposition.]