The answers to each of these questions are based on solving optimal control problems.

1. Suppose consumers choose consumption ©, fraction of time worked (n) and savings (s) to

$$\text{maximize } \int_{t=0}^{\infty} \frac{C_t^{1-\theta}(1-n_t)^{b(1-\theta)}}{1-\theta} e^{-\beta t} dt$$

subject to the constraint on savings that

$$\frac{ds_t}{dt} = (1-\tau)(w_t n_t + r_t s_t) - C_t$$

where τ is the income tax rate and w is the real wage rate. The variables r and w are exogenous to the consumer, and each parameter is a positive constant. Note that in the steady state, the fraction of time worked can not be rising or falling (i.e. must be constant).

A. What are the state and control variables?

B. Set up the Hamiltonian.

C. Derive the first-order necessary and transversality conditions?

D. Assuming that long-run per-capita consumption grows at the same rate as productivity, which is equal to g from our Solow model, what is the steady-state real interest rate? How does the tax rate affect effect r in the steady state?

E. Prove that if τ and w are held fixed, then C and n must always move in opposite directions.

2. Suppose the representative firm chooses investment (I) and employment (N) to maximize the present value of after-tax profits, where profits are defined as real revenues (Y) less the real costs of labor and investment spending, and $\tau$ represents the tax rate on a dollar of firm profits:

$$\int_{t=0}^{\infty} e^{-rt}(1-\tau)\left[Y_t - \left(\frac{W}{P}\right)_t N_t - I_t\right]dt\ .$$

Note r is a constant real rate of interest and $(W/P)_t$ is the real wage. Assume profits are maximized subject to a Cobb-Douglas production technology:

$$Y_t = K_t^{\alpha}(A_t N_t)^{1-\alpha}$$

and a capital (K) accumulation equation where $\delta$ is the rate of depreciation:

$$\dot{K}_t + \delta K_t = I_t\ .$$

A. What are the state and control variables?

B. Set up the Hamiltonian for this optimization problem.

C. Provide all the mathematical conditions necessary for the solution to this optimization problem.

D. Show and Explain whether or not this model implies that the marginal product of labor is equal to the real wage.

E. Show and Explain whether or not this model implies that the marginal product of capital is equal to the cost of capital (note that this problem has implicitly assumed there are no investment taxes, instead there is merely a corporate profits tax, and it also has assumed that the relative price of capital is 1).

3. This question models variables in per-effective-labor units. In the infinite horizon economy, the representative consumer chooses c to maximize:

$$\gamma \int_{t=0}^{\infty} e^{-\beta t} \left( \frac{c_t^{1-\theta}}{1-\theta} \right) dt ,$$

$$\left( \quad \text{where } \gamma = \frac{A_0^{1-\theta} L_0}{H} \quad \text{and} \quad \beta = \rho - n - (1-\theta) g \quad \right)$$

subject to the wealth constraint:

$$\dot{a}_t = w_t - c_t + a_t (r_t - g - n) - \tau_c c_t - \tau_i (w_t + a_t r_t) .$$

Consumption is taxed at the rate of $\tau_c$ and consumer income, $w_t + a_t r_t$ , is taxed at the rate $\tau_i$.

Assume for convenience that there is a single representative firm that chooses k to maximize after-tax profits:

$$f(k_t) - w_t - (r_t + \delta) k_t - \tau_k f(k_t)$$

and the firm pays tax based on its revenue at the rate of $\tau_k$.

Assume we use the standard national income accounting identity to construct the equation describing the time derivative of k:

$$\dot{k}_t = f(k_t) - c_t - (g + n + \delta) k_t - G_t \ .$$

Each tax rate is greater than or equal to zero and less than or equal to one:

$$\tau_j \in [0,1] \ \text{ for j=c,i,k.}$$

A. Set up the consumer optimization problem and obtain all conditions that must hold.

B. Set up the firm optimization problem and obtain all conditions that must hold.

C. For this competitive equilibrium, derive an expression for the time derivative of c in terms of k and possibly other variables (but not costate variables or Lagrange multipliers).

D. Provide the graphical model that determines the steady state values of k and c. You must be precise about which equations are being use used in the graph.

E. Make a copy of the graph from part D, then use it to show and explain what happens to $c_*$, $k_*$ and $c_0$ when G unexpectedly rises and this spending is financed by an increase in $\tau_i$. Your explanation will consist of a statement that a particular variable rises, falls, stays the same, or that the effect is ambiguous if more than one of these outcomes is possible.

F. Set up the Social Planner problem, and obtain all the conditions that must hold for this optimization problem.

G. Show whether or not the market equilibrium for variables (again not the co-state variable) is the same or different from the Social Planner outcome.