Solution to questions assigned in class:

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Homework: Keynes speculated that there was at best a doubling of the world standard of living in the 4000 years that ended in 1700. What is the annual growth rate if it takes 4000 years for GDP per capita to double?

Solution: From "the rule of 70": $70/g = 4000$

So $g = 70/4000 = .0175$ percent per year

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Homework: Using the numbers from the previous Table (Table 1.2 from Weil's text) calculate Poorland's GDP in terms of Richland Dollars based on the PPP exchange rate. Does the ratio of GDP for these two countries make sense?

Solution: The PPP exchange rate is the same as calculated in class:

30 Richland Dollars = 20 Poorland Dollars

Use this ratio to convert one currency into the other
20 Poorland Dollars * (30/20) Richland to Poorland Dollars equals 30 Richland Dollars

30 is 1/4 of 120, and 120 the GDP per capita of Richland when measured in Richland Dollars. This makes sense because Poorland produces 1/4 as much of each good as Richland.

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Homework: Use the two rules for taking derivatives to prove that: (A) the derivative of a constant is zero and (B) the derivative of a linear equation is equal to the slope of that line.

Solution:

(A) Let the constant be equal to a. You can write the constant as:

$a = ak^0$ because anything raised to the power zero equals 1

Using the power rule for derivatives, $\frac{da}{dk} = 0 \cdot a \cdot k^{-1}$ which equals 0, proving that the derivative of a constant is equal to zero (as long as k is not equal to zero

(B) The equation for a line is: $y = a + bk$. Calculate the derivative of y with respect to k to show that this derivative equals the slope of the line:

$$\frac{dy}{dk} = \frac{d}{dk}\left[a + bk\right] = \frac{d}{dk}\left[a\right] + \frac{d}{dk}\left[bk\right] = 0 + 1 \cdot b \cdot k^{1-1} = b$$

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Homework: Assume a Cobb-Douglas production function: $Y = A \cdot K^{\alpha} \cdot L^{1-\alpha}$ with $0 < \alpha < 1$. Suppose the economy starts at some initial levels of capital, labor and output: K_0, L_0 and Y_0. After a year goes by the levels of capital, labor and output become: K_1, L_1 and Y_1. Assume K_1 and L_1 each have grown by the same rate g from their initial levels:

$$K_1 = (1+g)K_0 \quad \text{and} \quad L_1 = (1+g)L_0$$

How is Y_1 related to Y_0? Does this production function have constant returns to scale?

Solution: Using the production function write Y_0 as a function of K_0 and L_0 :

$$Y_0 = A \cdot K_0^{\alpha} \cdot L_0^{1-\alpha}$$

and Y_1 as a function of K_1 and L_1 :

$$Y_1 = A \cdot K_1^{\alpha} \cdot L_1^{1-\alpha}$$

In this last equation insert $K_1 = (1+g)K_0$ and $L_1 = (1+g)L_0$:

$$\begin{aligned} Y_1 &= A \cdot [(1+g)K_0]^{\alpha} \cdot [(1+g)L_0]^{1-\alpha} \\ &= A \cdot (1+g)^{\alpha} K_0^{\alpha} \cdot (1+g)^{1-\alpha} L_0^{1-\alpha} \\ &= A \cdot K_0^{\alpha} \cdot L_0^{1-\alpha} (1+g)^{\alpha+1-\alpha} \\ &= A \cdot K_0^{\alpha} \cdot L_0^{1-\alpha} (1+g) \\ &= Y_0 (1+g) \end{aligned}$$

Therefore, Y_1 has increased by the same factor as K and L increased. This proves that the Cobb Douglas production function has constant returns to scale.

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Homework: Assume firms maximize profits as defined in class using a Cobb-Douglas production function. Now assume they chose K to maximize profits. Determine the first order condition. Show that the second order condition for a maximum is satisfied. Solve for the profit maximizing level of capital as a function of economic factors. Explain how each factor affects the demand for capital. Explain why these results make sense or why they do not make sense

Solution: Profits $= AK^{\alpha}L^{1-\alpha} - \left(\frac{W}{P}\right)L - uc \bullet K$.

Take the derivative of profits with respect to k:

$$\frac{dProfits}{dK} = \alpha \cdot A \cdot K^{\alpha-1}L^{1-\alpha} - uc$$

The second order condition comes from taking a second derivative of profits is:

$$\frac{d^2Profits}{dK^2} = \frac{d}{dK}\left[\frac{dProfits}{dK}\right] = (\alpha\text{-}1) \cdot \alpha \cdot A \cdot K^{\alpha-2}L^{1-\alpha}$$

All terms are positive except for (α-1), which is negative, making the second derivative negative. That means the first order condition obtains the maximum.

Setting the derivative equal to zero yields the first order condition which can be written as:

$$uc=\alpha \cdot A \cdot K^{\alpha-1}L^{1-\alpha}$$

Solve for K:

$$K=\frac{\alpha^{1/(1-\alpha)} \cdot A^{1/(1-\alpha)} \cdot L}{uc^{1/(1-\alpha)}}$$

K is positive and A and L have a positive effect on capital demand. And the uc has a negative effect. This makes sense because an increase in A or L will make capital more productive and so more attractive to firms and an increase in uc makes capital more expensive and so less attractive to firms.

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Homework: Show that if every unit of capital is paid the marginal product of capital and every unit of labor is paid the marginal product of labor, then the labor share of income and the capital share of income must sum to 1.

Hint: Use the first order conditions for choosing K and L.

Solution: If K is payed the MPK and L is payed the MPL then we have:

$$\text{MPK}\bullet\text{K} + \text{MPL}\bullet\text{L} = \left[\alpha \cdot A \cdot K^{\alpha-1}L^{1-\alpha}\right] \cdot K \; + \; \left[(1-\alpha) \cdot A \cdot K^{\alpha}L^{-\alpha}\right] \cdot L$$

$$= \alpha \cdot A \cdot K^{\alpha}L^{1-\alpha} \; + \; (1-\alpha) \cdot A \cdot K^{\alpha}L^{1-\alpha} = \alpha Y + (1-\alpha)Y$$

Where this result comes from the production function: $Y = A \cdot K^{\alpha}L^{1-\alpha}$

Thus, we see: MPK•K + MPL•L =Y. Then divide this equation by Y:

$$\frac{\text{MPK}\bullet\text{K}}{Y} + \frac{\text{MPL}\bullet\text{L}}{Y} = 1$$

And this last equation proves that the income shares for K and L sum to 1.

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