Solutions to Homework questions on: The Malthusian Model with Physical Capital and Productivity Growth

A. Divide the production function by L, and then simplify:

$$y = \frac{Y}{L} = \frac{K^{\alpha} Land^{\theta} (AL)^{1-\alpha-\theta}}{L} = k^{\alpha} \left(\frac{Land}{L}\right)^{\theta} (A)^{1-\alpha-\theta}.$$

B. See lecture slides for rules for the growth rate of a product, growth rate of a ratio and the growth rate of a variable raised to a power. Of course, the growth rate of a constant, like land, is zero. From these results it is straight-forward to show: $g_y = \alpha g_k - \theta g_L + (1-\alpha-\theta) g_A$

C. If a ratio is constant, our growth rule for the ratio shows that the numerator and denominator must each grow at the same rate: $g_y = g_k$

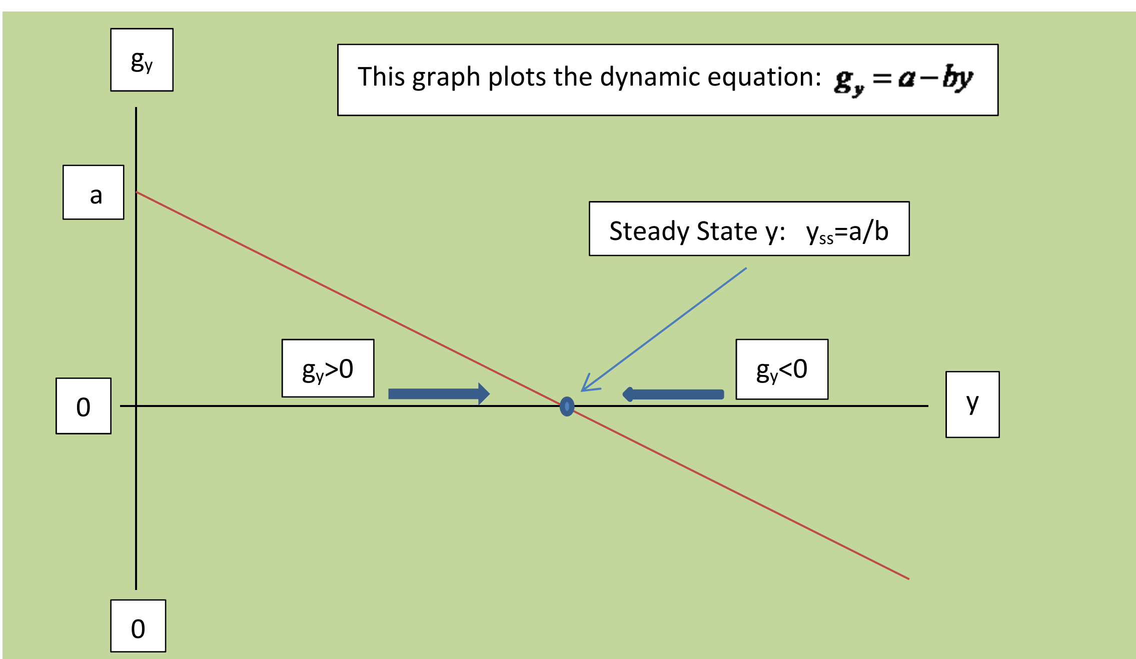
D. Use C to eliminate g_k from the result in B, then solve for g_y: $g_y = \left(\frac{-\theta}{1-\alpha}\right) g_L + \left(\frac{1-\alpha-\theta}{1-\alpha}\right) g_A$

E. Inserting $g_L = \beta_0 + \beta_1 y$ into result for part D, yields:

$$g_y = \left(\frac{1-\alpha-\theta}{1-\alpha}\right) g_A - \left(\frac{\theta\beta_0}{1-\alpha}\right) - \left(\frac{\theta\beta_1}{1-\alpha}\right) y$$

This is just a little more intricate version of the model we worked with in class. To see this point, write this equation as: $g_y = a - by$, where $a = \left[\left(\frac{1-\alpha-\theta}{1-\alpha}\right) g_A - \left(\frac{\theta\beta_0}{1-\alpha}\right)\right]$ and $b = \left(\frac{\theta\beta_1}{1-\alpha}\right)$.

The following graph "proves" that a steady state is achieved for y.



Since y_{ss}=a/b, insert the expressions for a and b to obtain: $y_{ss} = \left(\frac{(1-\alpha-\theta)g_A - \theta\beta_0}{\theta\beta_1} \right)$. This form of the solution is useful for easily seeing that an increase in β_1 causes y_{ss} to fall (because the numerator has to be positive for y_{ss} to be positive, which it must be). This makes sense - as people increase family size in response to income, there are more people and so output per capita falls. The solution for y_{ss} can also be written as: $y_{ss} = \left(\frac{(1-\alpha-\theta)g_A}{\theta\beta_1} \right) - \left(\frac{\beta_0}{\beta_1} \right)$ which allows us to easily determine the effect of all other parameters. In particular, a larger g_a clearly causes an increase in y_{ss} (because $(1-\alpha-\theta) > 0$). However, this is merely the **level not the growth rate** that increases – in contrast to the Solow model! Also, higher values of β_0, α or θ will reduce y_{ss}. The first of these means an increased birth rate reduces y_{ss}, which is what our textbook model also found. The other two results say that increase in either parameter from the production function will reduce y_{ss}.

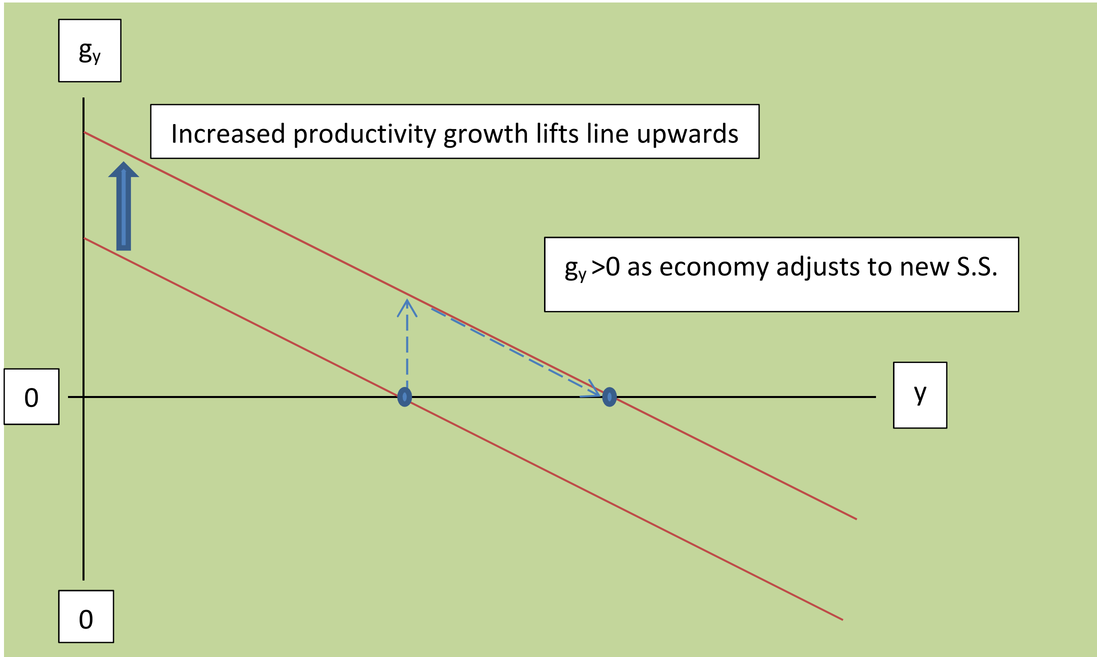
(You could also take a derivative to show each of these effects, if you wanted. But it's always easier, if possible, to examine an equation and use algebra to figure out the effect of a parameter, rather than take a derivative)

F. The steady state value of population growth comes from putting the result from part E into the population growth rate equation:

$$g_L^{ss} = \beta_0 + \beta_1 y_{ss} = \beta_0 + \beta_1 \left[\left(\frac{(1-\alpha-\theta)g_A}{\theta\beta_1} \right) - \left(\frac{\beta_0}{\beta_1} \right) \right] = \left(\frac{(1-\alpha-\theta)}{\theta} \right) g_A$$

We see that a larger g_A causes an increase in g_L^{ss} while higher values of either α or θ will cause a reduction in g_L^{ss}. These effects stem directly from their effect on y_{ss}. Interestingly, in this model the parameters from the equation relating family size to income per capita (β_0 and β_1) do not affect the steady state population growth rate. (This is a result of our using a linear population growth equation, not a general result).

G. An increase in productivity growth raises the parameter a, and that shifts the entire curve up the same amount at each point on the curve. Output growth is initially faster, but that rate eventually slows as the economy goes to the new y_{ss} which is higher than the initial steady state level. The rise in y causes population growth to also rise to a new higher steady state.



Output per-capita growth will be temporarily positive and population growth will be permanently higher.

H. The answer is qualitatively the same as in Part G. The difference in the graph is that a decrease in θ make the line steeper and the vertical intercept higher, but we know that it will raise y_{ss}. Thus output per-capita growth will be temporarily positive until y reaches its new higher steady state and population growth will be permanently higher.

I. Yes it can! While productivity growth rises, y will grow slowly and population growth will rise even more. This is what we saw in the data

J. Yes it can! Land's share of income was falling, and so y will grow slowly while population growth begins to rise even more. Again this is what we observed in the data.

K. The earlier solution for y_{SS} has β_1 in the denominator, so dividing by zero makes the solutions infinite … we say that such a solution is undefined. Solving the model, given β_1=0, first note that now we have $g_L^{ss} = \beta_0$. Hence, productivity growth no longer affects population growth and population growth is now exogenous. Next, insert this equation into our result from Part D, (which is valid irrespective of β_1's value) and obtain: $g_y^{ss} = \left(\frac{1-\alpha-\theta}{1-\alpha}\right) g_A - \left(\frac{\theta}{1-\alpha}\right)\beta_0$. Note that in this case, y doesn't have a steady state, but g_y does! That is to say, in this case when there is productivity growth, y may exhibit "steady state growth" In fact, this result is much the same as a Solow model result (one that we may show later). In that model where productivity grows and Land is irrelevant to production: $g_y^{ss} = g_A$ .That is the precisely the same result we get from our last equation by setting θ = 0 in the analysis done here (which assumed β_1 = 0).