Toeplitz-composition algebras generated by piecewise quasicontinuous symbols and a linear fractional non-automorphism fixing a boundary point

Yi Yan

Abstract. Commutative Toeplitz-composition subalgebras of the Calkin algebra on the classical Hardy space are analyzed with piecewise quasicontinuous symbols and a linear fractional non-automorphism fixing a boundary point. For the parabolic case the fiber structure of the maximal ideal space of the C*-subalgebra is explicitly described, resulting in essential spectrum and essential norm formulas. For the non-parabolic case, both the maximal ideal space of and the Shilov boundary for a non-self-adjoint subalgebra are identified, allowing for a reasonable estimate of essential spectra. Certain Fredholm index equalities are also obtained. Our method is based on elementary theory of commutative Banach algebras.

Mathematics Subject Classification (2020). 47B35, 47B33, 47L80, 46J20.

Keywords. Toeplitz operator, composition operator, piecewise quasicontinuous, linear fractional, Shilov boundary, essential spectrum, Fredholm index.

1. Introduction and notations

Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$. $L^2$ and $L^\infty$ are the Lebesgue spaces of square integrable and essentially bounded measurable complex functions, respectively, on the unit circle $\partial \mathbb{D}$. $H^2 \subset L^2$ denotes the Hardy subspace, and $H^\infty \subset L^\infty$ the subalgebra of nontangential limits on $\partial \mathbb{D}$ of bounded analytic functions in $\mathbb{D}$. Let $C$, $PC$, and $QC := (H^\infty + C) \cap H^\infty + C$ be the C*-subalgebras of $L^\infty$ consisting respectively of continuous, piecewise continuous, and quasicontinuous functions on $\partial \mathbb{D}$. The C*-algebra generated by $PC$ and $QC$ is denoted by $PQC$ and consists of the so-called piecewise quasicontinuous functions.

For $f \in L^\infty$, $M_f \in \mathcal{L}(L^2)$ denotes the multiplication operator by $f$ on $L^2$ whose compression to $H^2$, $T_f := PM_f | H^2 \in \mathcal{L}(H^2)$, is the Toeplitz operator with

The author wishes to thank Professors Tyrone Duncan and Albert Sheu for their guidance and encouragement.
symbol $f$. Also, any analytic self-map $\phi$ of $\mathbb{D}$ defines a composition operator $C_\phi \in \mathcal{L}(H^2)$ (cf. [7]). For a subset $S \subset L^\infty$, $\mathcal{T}(S)$ denotes the $C^*$-subalgebra of $\mathcal{L}(H^2)$ generated by the Toeplitz operators with symbol in $S$, and $\mathcal{T}\mathcal{C}(S,\phi)$ the $C^*$-algebra generated by $\mathcal{T}(S)$ and $C_\phi$. Letting $T \in \mathcal{L}(H^2) \mapsto [T] \in \mathcal{L}(H^2) / \mathcal{K}(H^2)$ be the quotient map onto the Calkin algebra, it is known that (cf. [1])

\[
[T_f][T_g] = [T_g][T_f] = [T_{fg}], \quad f \in QC, \ g \in L^\infty
\]

\[
[T_f][T_g] = [T_g][T_f] \neq [T_{fg}], \quad f, \ g \in PQC.
\]

A large class of results on Toeplitz-composition $C^*$-algebras on $H^2$ involve linear fractional non-automorphisms (cf. [13]). Let $\phi$ be a non-automorphism, linear fractional self-map of $\mathbb{D}$ with $\phi(\zeta) = \eta$ for a pair $\zeta, \eta \in \partial \mathbb{D}$ (that is, $C_\phi$ is not compact). Then the results in [10] on compact Toeplitz-weighted composition operators and the explicit formula [6] for the adjoint $C^*_\phi$ imply that continuous Toeplitz operators interact with $C_\phi$ like constant multiples, and that $C^*_\phi$ equals a constant multiple of a closely related composition, both modulo $\mathcal{K}(H^2)$. These key properties were first found and used in [11] to construct an isomorphism of the non-commutative $[\mathcal{T}\mathcal{C}(C,\phi)]$, $\phi(\zeta) = \eta \neq \zeta$, based on Allan-Douglas localization ([8, 1]), by which essential spectra of operators in $\mathcal{T}\mathcal{C}(C,\phi)$ were explicitly computed. On the other hand, for $\phi$ fixing $\zeta$, isomorphisms of $[\mathcal{T}\mathcal{C}(C,\phi)]$ were established in [12] depending on parabolicity of $\phi$. In particular, the isomorphism [12, Corollary 6.6] of the non-commutative $[\mathcal{T}\mathcal{C}(C,\phi)]$ for a non-parabolic $\phi$ involves a crossed product of continuous functions by a cyclic group action of infinite order, under which essential spectra remain elusive.

It seems natural to go beyond the continuous Toeplitz symbols. Piecewise continuous symbols were considered in [16], while quasicontinuous symbols have fundamentally different behavior. In fact, $PC \cap QC = C$ by Sarason’s characterization [14] of QC indicates they extend $C$ in different directions. This paper investigates Toeplitz-composition algebras generated by certain $PQC$ symbols and a $\phi$ fixing a boundary point. For the parabolic case in Section 3, we consider the commutative Calkin $C^*$-subalgebra of the generators and obtain the complete fiber structure of its maximal ideal space. This is achieved in part by leveraging Sarason’s work [15], and in part by a direct-sum decomposition using partial knowledge of the maximal ideals which then uncovers the complete structure. For the non-parabolic case in Section 4, however, we consider a commutative non-self-adjoint Calkin subalgebra instead. The maximal ideal space is similarly described in terms of the polynomial-convex hull of the essential spectrum of $C_\phi$, and the fairly large Shilov boundary is identified on the basis of the Toeplitz $C^*$-subalgebra and the abundance of singleton fibers. On spectral permanence considerations, reasonable estimates for essential spectra are obtained as a result of these identifications.

There are several reasons for the choice of the commutative non-self-adjoint algebra in lieu of the larger non-commutative $C^*$-algebra for the non-parabolic case. The crossed-product approach as in [12] seems intractable due to the elevated complexity of the $PQC$ functions. The localization approach used in [11]
derives its effectiveness from the availability of a large central $C^*$-subalgebra containing a crucial element generated by $C_\phi$ and $C_\phi^*$ due to $[C_\phi]^2 = 0$. For $\phi$ fixing a boundary point, this critical feature disappears and the localization approach seems in doubt. On the flip side, one does lose the ability to bring $C_\phi^*$ in the combination for spectral analysis, in contrast to the $C^*$-algebraic approaches.

We end this section with additional notations. The spectrum of an element $a$ of a unital Banach algebra $U$ is denoted by $\sigma(a)$, or more specifically $\sigma(a,U)$ when subalgebras are in the context, and the spectral radius by $\rho(a)$. The maximal ideal space $M(A)$ of a commutative unital Banach algebra $A$ is equipped with the Gelfand topology, and the fiber $M_\alpha(A)$ consists of the extensions in $M(A)$ of $x \in M(B)$ for a subalgebra $B$. $C(\Omega)$ stands for the commutative $C^*$-algebra of continuous complex functions on a compact Hausdorff space $\Omega$, while $C$ is reserved for that on $\partial \mathbb{D}$. For an operator $T$, the essential norm and spectrum are $\|T\|_e := \|[T]\|$, $\sigma_e(T) := \sigma([T])$, relative to the Calkin algebra.

2. Preliminary facts independent of parabolicity

We consider Toeplitz-composition algebras on the Hardy space $H^2$ generated by Toeplitz symbols in a subalgebra of $PQC$ and a composition symbol $\phi$ being a non-automorphism, linear fractional self-map of $\mathbb{D}$ with $\|\phi\|_\infty = 1$. Such $\phi$ must satisfy $\phi(\zeta) = \eta$ for a unique pair $\zeta, \eta \in \partial \mathbb{D}$, and the Krein adjoint (cf. [7]) $\psi$ of $\phi$ is of the same type with $\psi(\eta) = \zeta$. If $\zeta = \eta$ and $\phi'(\zeta) = 1$, $\phi$ is parabolic and takes the explicit form (cf. [2])

$$\phi(z) = \frac{(2 - \alpha)z + \alpha \zeta}{-\alpha \zeta z + (2 + \alpha)}$$

where $\alpha, \Re \alpha > 0$, is called the translation number of $\phi$. Otherwise, $\phi$ is non-parabolic. In either case, $|\phi| < 1$ everywhere on $\partial \mathbb{D}$ except $\zeta$, and the set $E(\phi)$ associated with the Aleksandrov-Clark measures of $\phi$ is the singleton $\{\zeta\}$. Thus, the following lemma is a specialization to $\phi$ of [11, Corollary 2.2].

**Lemma 2.1.** Let $\phi$ be a non-automorphism, linear fractional self-map of $\mathbb{D}$ satisfying $\phi(\zeta) = \eta$ for some $\zeta, \eta \in \partial \mathbb{D}$. Then for every $f \in L^\infty$ continuous at $\zeta$, $T_fC_\phi$ is compact if and only if $f(\zeta) = 0$.

It follows that for every $f \in L^\infty$ continuous at both $\zeta$ and $\eta$,

$$[T_f][C_\phi] = f(\zeta)[C_\phi], \quad [T_f][C_\psi] = \overline{f(\eta)}[C_\psi].$$

Taking adjoints of the latter equality yields $[C_\psi^*][T_f] = f(\eta)[C_\psi^*]$. Since $[C_\psi^*] = s[C_\psi]$ and $s[C_\psi^*] = [C_\phi]$ where $s = |1/\phi'(\zeta)| > 0$ [11, Theorem 3.1, 3.6], one has exactly as in [11] that

$$[T_f][C_\phi] = f(\zeta)[C_\phi], \quad [C_\phi][T_f] = f(\eta)[C_\phi].$$

Recall that a function $f \in L^\infty$ is said to be continuous at $\zeta \in \partial \mathbb{D}$ with $f(\zeta) = c$ if $\lim_{\epsilon \to 0} \|f|((\zeta - \epsilon, \zeta + \epsilon) - c\|_\infty = 0$, or equivalently $f|M_{\zeta}(L^\infty) \equiv c$ (cf.
2.79(b)), \( \zeta \in \partial \mathbb{D} \cong M(C) \). Here of course \((\zeta - \epsilon, \zeta + \epsilon)\) is a subarc of the circle centered at \( \zeta \). Set
\[
PQC(\zeta) := \{ f \in PQC : f \text{ is continuous at } \zeta \}.
\]
the C*-subalgebra of functions in \( PQC \) whose Gelfand transform is constant on the fiber \( M_\zeta(PQC) \).

This paper exclusively concerns those \( \phi \) fixing \( \zeta \), so that the key relations above yield
\[
[T_f][C_\phi] = [C_\phi][T_f] = f(\zeta)[C_\phi], \quad f \in PQC(\zeta).
\]
It is known that \( [\mathcal{T}(PQC)] \) is commutative. Also, \( C_\phi \) is essentially normal if and only if \( \phi \) is parabolic \([2] [11]\). Thus it follows from (2.1) that the C*-algebra \( \mathcal{T}(PQC(\zeta), \phi) \) is commutative if and only if \( \phi \) is parabolic, while the non-self-adjoint Calkin subalgebra generated by \( PQC(\zeta) \) and \( \phi \) is commutative regardless.

Now we are ready to derive some preliminary results common to the next two sections. First we need to obtain an explicit description of the maximal ideal space \( M([\mathcal{T}(PQC(\zeta))] \) of the commutative Toeplitz C*-subalgebra. This is done by describing its fiber structure over the circle, on the basis of Sarason’s work \([15]\) on that of \( M([\mathcal{T}(PQC)]) \). The key to the leverage on Sarason’s result lies in a lemma under the following setup: Let
\[
r_1 : M([\mathcal{T}(PQC)]) \rightarrow M([\mathcal{T}(PQC(\zeta))] \text{ and } n_2 : M([\mathcal{T}(PQC(\zeta))] \rightarrow M([\mathcal{T}(C)])
\]
be the surjective restriction maps. Then, \( n_2 \circ r_1 \) is the restriction map from \( M([\mathcal{T}(PQC)]) \) onto \( M([\mathcal{T}(C)]) \). Here \( M([\mathcal{T}(C)]) \cong M(C) \) under the isomorphism \( f \in C \mapsto [T_f] \in \mathcal{T}(C) \), and
\[
r_1(M_\lambda([\mathcal{T}(PQC)])) = n_2^{-1}(\lambda) = M_\lambda([\mathcal{T}(PQC(\zeta)])], \quad \forall \lambda \in \partial \mathbb{D} \cong M(C).
\]

**Lemma 2.2.** The map \( r_1 \) on the fiber \( M_\zeta([\mathcal{T}(PQC)]) \) is a constant, denoted by \( \langle \zeta \rangle \) and determined by
\[
\langle \zeta \rangle([T_f]) = f(\zeta), \quad f \in PQC(\zeta).
\]
For every \( \lambda \neq \zeta \in \partial \mathbb{D} \), the map \( r_1 \) on the fiber \( M_\lambda([\mathcal{T}(PQC)]) \) is a homeomorphism onto \( M_\lambda([\mathcal{T}(PQC(\zeta)])\).

**Proof.** Let \( \xi \in M_\zeta([\mathcal{T}(PQC)]) \), so that \( \xi([T_\zeta]) = \zeta \). It suffices to prove the first part by showing \( \xi([T_f]) = f(\zeta) \) for every \( f \in PQC(\zeta) \). Consider the Gelfand transform of \( \phi(z) := z - \zeta \) on \( M(PQC) \) with zero set precisely \( M_\zeta(PQC) \). Then \( f \in PQC(\zeta) \) implies that the zero set of \( f - f(\zeta) \) on \( M(PQC) \) contains \( M_\zeta(PQC) \) which is also the zero set of the closed principal ideal \( \phi PQC \) in \( PQC \). Thus \( f - f(\zeta) \in \phi PQC \) by radicality of the ideals in \( PQC \). Hence, for every \( \epsilon > 0 \), there exists \( g \in PQC \) with \( \|f - f(\zeta) - \phi g\|_\infty \leq \epsilon \). It follows that
\[
|\xi([T_f]) - f(\zeta)| = |\xi([T_f - f(\zeta)])| \leq |\xi([T_\phi])\xi([T_g])| + \epsilon
\]
\[
= |\xi([T_\zeta]) - \zeta|\xi([T_g])| + \epsilon = \epsilon.
\]
So, \( \xi([T_f]) = f(\zeta) \) as required.
To prove the second part, let \( g \in C \) with \( g(\lambda) = 1, g(\zeta) = 0 \). For \( \xi_1 \neq \xi_2 \in M_\lambda([T(PQC)]) \), there must exist \( f \in PQC \) such that \( \xi_1([T_f]) \neq \xi_2([T_f]) \). Consider the product function \( gf \). Evidently, \( gf \in PQC(\zeta) \) with \( (gf)(\zeta) = 0 \). Since for \( k = 1, 2 \)
\[
r_1(\xi_1)([T_{gf}]) = \xi_1([T_{gf}]) = \xi_1([T_f])\xi_1([T_f]) = g(\lambda)\xi_1([T_f]) = \xi_1([T_f]),
\]
one has \( r_1(\xi_1) \neq r_1(\xi_2) \). That is, the continuous map \( r_1 \) on \( M_\lambda([T(PQC)]) \) is injective and hence a homeomorphism onto \( M_\lambda([T(PQC)(\zeta)]) \).

The fibers of \( M([T(PQC(\zeta))]) \) over \( \partial \mathbb{D} \) are described by the following theorem whose proof is immediate from Sarason’s result ([15] Lemma 14, 15, 16; cf. [4] 4.87]), (2.2), and the preceding lemma. Some notations need to be explained. For \( \lambda \in \partial \mathbb{D} \), the closed subsets \( M_\lambda^+(QC) \) of \( M(QC) \) are defined to be respectively the zero set of the closed ideals \( \{ f \in QC : f(\lambda \pm) = 0 \} \) of \( QC \), \( M_\lambda(QC) = M_\lambda^+(QC) \cup M_\lambda^-(QC) \) with \( M_\lambda^0(QC) := M_\lambda^+(QC) \cap M_\lambda^-(QC) \neq \emptyset \). Sarason showed that ([15], Lemma 13 and surrounding remarks) \( M_y(PQC) \) over \( y \in M_\lambda^+(QC) \setminus M_\lambda^0(QC) \) consists of a single functional \( y_\pm \) whose action on \( f \in PC \) gives \( f(\lambda \pm) \), while \( M_y(PQC) \) over \( y \in M_\lambda^0(QC) \) consists of two distinct functionals \( y_+, y_- \), again determined by \( f(y_\pm) = f(\lambda \pm), f \in PC \).

**Theorem 2.3.** The fiber \( M_\zeta([T(PQC(\zeta))]) \) consists only of the functional \( \langle \zeta \rangle \). For every \( \lambda \neq \zeta \in \partial \mathbb{D} \) the fiber \( M_\lambda([T(PQC(\zeta))]) \) is the disjoint union of a family \( \{ F_y : y \in M_\lambda(QC) \} \) of closed subsets, where \( F_y \) for \( y \in M_\lambda^+(QC) \setminus M_\lambda^0(QC) \) consists of a single functional assuming the value \( f(y_\pm) \) at \([T_f] \) for every \( f \in PQC(\zeta) \), and where \( F_y \) for \( y \in M_\lambda^0(QC) \) is homeomorphic to \([0, 1] \) via the *-isomorphism between \( C(F_y) \) and \( C[0, 1] \) determined by
\[
[T_f]F_y \mapsto (t \in [0, 1] \mapsto tf(y_+) + (1 - t)f(y_-)), \quad f \in PQC(\zeta).
\]

The next lemma extends [2.1] to operators in \( T(PQC(\zeta)) \). Note in passing that the Toeplitz symbol map [3] Chapter 7 \( \tau : T(L^\infty) \to L^\infty \) takes \( T(PQC(\zeta)) \) to \( PQC(\zeta) \) and satisfies \( \tau(T)(\zeta) = \langle \zeta \rangle([T]) \), \( T \in T(PQC(\zeta)) \).

**Lemma 2.4.** For every \( T \in T(PQC(\zeta)) \),
\[
[T][C_\phi] = [C_\phi][T] = \langle \zeta \rangle([T])[C_\phi].
\]

**Proof.** If \( T \) is a finite product of Toeplitz operators with \( PQC(\zeta) \) symbols, then the equalities follow from (2.1) and (2.3). One completes the proof by passing to sums of products and then to the closure by continuity. \( \square \)

The remainder of this section contains some general considerations for commutative Banach algebras \( A \). A closed subset \( F \subset M(A) \) is called a boundary for \( A \) if \( \| \hat{a}F \|_\infty = \| \hat{a} \|_\infty \) for the Gelfand transform \( \hat{a} \in C(M(A)) \) of every \( a \in A \). The intersection of all the boundaries for \( A \) is again a boundary, \( \partial A \), called the Shilov boundary (cf. [9]). The Shilov boundary plays a role in the following extension result on maximal ideals, which is a somewhat stronger version of Shilov’s original \( r(M(A)) \supseteq \partial B \). A proof is provided since a reference is not found.
Proposition 2.5. Let $B$ be a closed subalgebra of a commutative Banach algebra $A$, and let $r : M(A) \to M(B)$ be the restriction map. Then $r(\partial A) \supset \partial B$.

Proof. Since $r(\partial A)$ is a closed subset of $M(B)$, it suffices to show that every $\tilde{b} \in C(M(B))$, $b \in B$, attains on $r(\partial A)$ its maximum modulus $\rho(b, B)$. But this follows from the relation 

$$\tilde{b} = \hat{b} \circ r$$

where the Gelfand transform $\tilde{b} \in C(M(A))$ of $b \in B \subset A$ attains on $\partial A$ its maximum modulus $\rho(b, A) = \rho(b, B)$.

Let $U$ be a non-commutative unital Banach algebra, and $A(a)$ be the closed subalgebra generated by $a \in U$. It is well known that

$$\sigma(a, A(a)) = \text{hull}(\sigma(a, U)), \quad (2.4)$$

where the polynomial-convex hull equals the complement of the unbounded component of that of $\sigma(a, U)$. Let $B$ be a commutative closed subalgebra of $U$,

$$\sigma(b, U) \subset \sigma(b, B) = \hat{b}(M(B))$$

for $b \in B$. Taking $A$ to be either the double commutant algebra of $B$ or a maximal commutative subalgebra of $U$ containing $B$, one has by spectral permanence

$$\sigma(b, U) = \sigma(b, A) = \hat{b}(M(A)) = \hat{b}(r(M(A))) \supset \hat{b}(\partial B)$$

for $b \in B \subset A$ with Gelfand transform $\tilde{b}$ on $M(A)$ and $r : M(A) \to M(B)$ as before. Combining the two inclusions gives

$$\hat{b}(\partial B) \subset \sigma(b, U) \subset \hat{b}(M(B)), \quad \forall b \in B. \quad (2.5)$$

This relation can be useful if $\partial B$ is relatively large in $M(B)$. Proposition 2.5 contributes to the explicit identification of $\partial B$ for a certain commutative Calkin subalgebra $B$ in the non-parabolic case.

3. The parabolic case

Assume in this section $\phi$ is parabolic, that is, $\phi'(\zeta) = 1$ at the fixed boundary point $\zeta$. Then one has a commutative C*-subalgebra $[TC(PQC(\zeta), \phi)]$ of the Calkin algebra. The goal is to obtain an explicit description of its maximal ideal space, so that the essential spectrum and norm of certain operators of interest can be computed, and that the Fredholm index can be determined.

First note the essential spectrum $\sigma_e(C_\phi)$ has an explicit form. For, conjugating with a rotation, we may take

$$\phi(z) = \frac{(2 - \alpha)z + \alpha}{-\alpha z + (2 + \alpha)}, \quad \phi(1) = 1$$

where $\alpha, \Re \alpha > 0$, is the original translation number of $\phi$. By the half-plane version ($\theta = \pi/2$) of a result of C. Cowen ([7]; cf. [7, Corollary 7.42]), $\sigma(C_\phi)$ is a logarithmic spiral in the disc from 1 to 0

$$\sigma(C_\phi) = \{e^{-\alpha t} : t \geq 0\} \bigcup \{0\} =: e^{-\alpha [0, \infty]}.$$
Since such a spiral has empty interior in $C$ and no isolated points, [4, Theorem 37.8] asserts that
\[ \sigma_e(C_\phi) = \sigma(C_\phi) = e^{-\alpha[0,\infty]} . \] (3.1)
Note that $\sigma_e(C_\phi)$ is homeomorphic to $[0,1]$ via the modulus map $z \mapsto |z|$.

In view of Theorem 2.3, it remains to find the fibers of $M([T\mathcal{C}(PQC(\zeta), \phi)])$ over $M([T\mathcal{C}(PQC(\zeta))]).$ The fibers over $\xi \neq \langle \zeta \rangle \in M([T\mathcal{C}(PQC(\zeta))])$ are singletons.

**Theorem 3.1.** For each $\xi \neq \langle \zeta \rangle \in M([T\mathcal{C}(PQC(\zeta))]),$ the fiber $M_\xi([T\mathcal{C}(PQC(\zeta), \phi)])$ consists of a single functional vanishing at $[C_\phi].$

*Proof.* Since $\xi \neq \langle \zeta \rangle, \lambda := \xi|[T(C)] \neq \zeta \in \partial D$ and $\xi([T_g]) = g(\lambda)$ for $g \in C$. Choose $g \in C$ with $g(\zeta) = 1$ while $g(\lambda) = 0.$ Applying $x \in M_\xi([T\mathcal{C}(PQC(\zeta), \phi)])$ on $\mu$ with $f = g$ gives $x([C_\phi]) = 0.$ Since the functionals in the nonempty fiber are determined by their common value at the $C^*$-generator $[C_\phi],$ the assertion follows at once. \[ \square \]

The fiber $M_{\langle \zeta \rangle}([T\mathcal{C}(PQC(\zeta), \phi)])$ will be identified after several lemmas. Consider the isomorphism $f \in C(e^{-\alpha[0,\infty]}) \leftrightarrow f([C_\phi]) \in C^*([C_\phi])$ via the continuous functional calculus for the essentially normal operator $C_\phi.$ Write
\[ C_0(e^{-\alpha[0,\infty]}) := \{ f \in C(e^{-\alpha[0,\infty]}): f(0) = 0 \}, \]
\[ C_0([C_\phi]) := \{ f([C_\phi]): f \in C_0(e^{-\alpha[0,\infty]}). \}

**Lemma 3.2.** For every $T \in \mathcal{T}(PQC(\zeta))$ and $a \in C_0([C_\phi]),$
\[ [T]a = a[T] = \langle \zeta \rangle([T])a. \]

*Proof.* Since the maximal ideal $C_0(e^{-\alpha[0,\infty]})$ is singly generated by the identity function, $f \in C_0(e^{-\alpha[0,\infty]})$ can be uniformly approximated by $z g, g \in C(e^{-\alpha[0,\infty]}).$ That is, $a = f([C_\phi])$ can be approximated in the Calkin algebra by $[C_\phi]g([C_\phi]) = g([C_\phi])[C_\phi].$ By Lemma 2.4,
\[ [T][C_\phi]g([C_\phi]) = \langle \zeta \rangle([T])[C_\phi]g([C_\phi]), \quad g([C_\phi])[C_\phi][T] = \langle \zeta \rangle([T])g([C_\phi])[C_\phi]. \]
The proof is complete after passing to the limit. \[ \square \]

**Lemma 3.3.** There exists $x \in M_{\langle \zeta \rangle}([T\mathcal{C}(PQC(\zeta), \phi)])$ such that $x([C_\phi]) = 0.$

*Proof.* Choose a sequence $\lambda_n \to \zeta$ on the circle, $\lambda_n \neq \zeta.$ Next choose $\xi_n \in M_{\lambda_n}([\mathcal{T}(PQC(\zeta))])$ for every $n.$ According to the previous theorem, let $x_n$ be the functional in $M_{\xi_n}([T\mathcal{C}(PQC(\zeta), \phi)])$ with $x_n([C_\phi]) = 0.$ The sequence $x_n$ in $M([\mathcal{T}(PQC(\zeta), \phi)])$ has a cluster point $x,$ and
\[ x_{n_\omega} \to x \]
for a subnet indexed by $\omega.$ Evidently, $x([C_\phi]) = 0.$ Applying the convergence on $[T_f], f \in C,$ yields $f(\lambda_n) \to x([T_f])$ while $f(\lambda_n) \to f(\zeta).$ Therefore, $x([T_f]) = f(\zeta)$ for every $f \in C,$ so
\[ x([T(PQC(\zeta))]) \in M_{\xi}([\mathcal{T}(PQC(\zeta))]) = \langle \zeta \rangle \]
by Lemma 2.2. Such $x$ fulfills the requirement. \[ \square \]
The next lemma uses an essential norm relation to express \([TC(PQC(\xi), \phi)]\) as the direct sum of its Toeplitz C*-subalgebra and a closed ideal of its composition C*-subalgebra.

**Lemma 3.4.** For every \(T \in T(PQC(\xi))\) and \(a \in C_0([C_\phi])\),
\[
\|T + a\| \geq \|T\|. \tag{3.2}
\]
Consequently, one has the Banach space direct-sum decomposition
\[
[TC(PQC(\xi), \phi)] = [T(PQC(\xi))] \bigoplus C_0([C_\phi]).
\]

**Proof.** Since the Gelfand transform of the C*-algebra \([TC(PQC(\xi), \phi)]\) is isometric, one has the representations
\[
\|T + a\| = \max\{|x(T)| + |x(a)| : x \in M([TC(PQC(\xi), \phi)])\}, \tag{3.3}
\]
\[
\|T\| = \max\{|x(T)| : x \in M([TC(PQC(\xi), \phi)])\}. \tag{3.4}
\]

We proceed by partitioning \(M([TC(PQC(\xi), \phi)])\) in fibers over \(M([T(PQC(\xi)])\).

As before, the fiber \(M_\xi([TC(PQC(\xi), \phi)])\) for any \(\xi \not\in \langle \xi \rangle \subset M([T(PQC(\xi)])\) consists of a single functional \(x\) vanishing at \([C_\phi]\). Thus, \([C_\phi]\) lies in the maximal ideal \(\ker(x) \cap C^*([C_\phi])\) of \(C^*([C_\phi])\). Under the continuous functional calculus for \([C_\phi]\), \(C_0([C_\phi])\) is the closed principal (and maximal) ideal of \(C^*([C_\phi])\) generated by \([C_\phi]\) because the function ideal is so. Thus, \(a \in C_0([C_\phi]) \subset \ker(x)\) and \(|x(T)| + |x(a)| = |x(T)|\) for such \(x\). For every \(x \in M_\xi([TC(PQC(\xi), \phi)])\), \(x([T]) = \langle \xi \rangle([T])\), while \(x_0([C_\phi]) = 0\), hence \(x_0(a) = 0\), for at least one \(x_0\) in this fiber by the previous lemma. Combining these two cases that exhaust \(M([TC(PQC(\xi), \phi)])\), the norm inequality follows from (3.3) and (3.4).

It immediately follows from (3.2) that \([T(PQC(\xi))] \cap C_0([C_\phi]) = \{0\}\) and that the sum is a norm-closed, self-adjoint linear subspace of \([TC(PQC(\xi), \phi)]\) containing all of its generators. The sum is also closed under multiplication, by Lemma 3.2, and is therefore a C*-subalgebra of the Calkin algebra containing \([TC(PQC(\xi), \phi)]\). The proof is complete. \(\Box\)

**Remark 3.5.** Incidentally, the two maximal ideals \(\ker(x) \cap C^*([C_\phi]) = C_0([C_\phi])\) for every \(x \in M([TC(PQC(\xi), \phi)])\) vanishing at \([C_\phi]\).

**Theorem 3.6.** The fiber \(M_\langle \xi \rangle([TC(PQC(\xi), \phi)])\) is homeomorphic to \(e^{-\alpha[0, \infty]}\) via the map
\[
x \in M_\langle \xi \rangle([TC(PQC(\xi), \phi)]) \mapsto x([C_\phi]) \in e^{-\alpha[0, \infty]}.
\]

**Proof.** The continuous map is certainly injective on the generator \([C_\phi]\) with range contained in \(\sigma_\phi(C_\phi) = e^{-\alpha[0, \infty]}\), by spectral permanence in the C*-subalgebra of the Calkin algebra. It remains only to show surjectivity. To this end, fix an arbitrary \(\lambda \in e^{-\alpha[0, \infty]}\) and consider the multiplicative linear functional
\[
v_\lambda : a = f([C_\phi]) \in C_0([C_\phi]) \mapsto f(\lambda).
\]
We claim that the direct-sum linear functional \(\langle \xi \rangle \bigoplus v_\lambda\) defined on
\[
[TC(PQC(\xi), \phi)] = [T(PQC(\xi))] \bigoplus C_0([C_\phi])
\]
is also multiplicative. For, given $T, S \in \mathcal{T}(PQC(\zeta))$ and $a = f([C_\phi]), b = g([C_\phi]) \in C_0([C_\phi])$,

$$([T] + a)([S] + b) = [T][S] + [T]b + a[S] + ab$$

$$= [T][S] + \langle \zeta \rangle ([T])b + \langle \zeta \rangle ([S])a + ab$$

by Lemma 3.2, so that

$$\langle \zeta \rangle \bigoplus v_\lambda(([T] + a)([S] + b))$$

$$= \langle \zeta \rangle ([T]) \langle \zeta \rangle ([S]) + \langle \zeta \rangle ([T])g(\lambda) + \langle \zeta \rangle ([S])f(\lambda) + f(\lambda)g(\lambda)$$

$$= \left( \langle \zeta \rangle ([T]) + f(\lambda) \right) \left( \langle \zeta \rangle ([S]) + g(\lambda) \right)$$

as desired. Therefore, $x := \langle \zeta \rangle \bigoplus v_\lambda \in M_{\langle \zeta \rangle}([\mathcal{T}C(PQC(\zeta), \phi)])$ satisfies $x([C_\phi]) = \lambda$, and the proof is complete.

Theorem 2.3, 3.1, and 3.6 together determine the behavior of cosets in $[\mathcal{T}C(PQC(\zeta), \phi)]$ on its maximal ideal space, from which essential spectrum and norm formulas are derived in the following theorem for certain combinations of Toeplitz and composition operators. For a bivariate polynomial $p(z, w) = \sum_{0 \leq j+k \leq n} \beta_{j,k} z^j w^k$, $\beta_{j,k} \in \mathbb{C}$, denote the operator

$$p(C_\phi, C_\phi^*) := \beta_{0,0}I + \sum_{1 \leq j+k \leq n} \beta_{j,k} \sum_{l=1}^{j+k} S_{j,k,l}$$

where each $(j + k)$-tuple $\{S_{j,k,l} : 1 \leq l \leq j + k\}$ is a permutation of $j$ occurrences of $C_\phi$ and $k$ of $C_\phi^*$. Note that for any $x \in M([\mathcal{T}C(PQC(\zeta), \phi)])$

$$x([p(C_\phi, C_\phi^*)]) = p(x([C_\phi]), x([C_\phi^*])).$$

(3.5)

Let $\langle \zeta, \lambda \rangle$ be the functional $x \in M_{\langle \zeta \rangle}([\mathcal{T}C(PQC(\zeta), \phi)])$ with $x([C_\phi]) = \lambda$. By Theorem 3.1,

$$M_0 := \bigcup_{\lambda \neq \zeta \in \partial \mathbb{D}} M_\lambda([\mathcal{T}C(PQC(\zeta), \phi)]) \bigcap \langle \zeta, 0 \rangle$$

$$= \{x \in M([\mathcal{T}C(PQC(\zeta), \phi)] : x([C_\phi]) = 0\}. \quad (3.6)$$

Theorem 3.7. For $f \in PQC(\zeta)$ and a bivariate polynomial $p$ with $p(0, 0) = 0$,

$$\sigma_e(T_f + p(C_\phi, C_\phi^*)) = \sigma_e(T_f) \bigcup \{f(\zeta) + p(\lambda, \bar{\lambda}) : \lambda \in e^{-\alpha[0, \infty]}\},$$

$$\|T_f + p(C_\phi, C_\phi^*)\|_e = \|f\|_\infty \lor \max\{\|f(\zeta) + p(\lambda, \bar{\lambda})\| : \lambda \in e^{-\alpha[0, \infty]}\},$$

$$\sigma_e(T_f) = \{f(\zeta)\} \bigcup_{\lambda \neq \zeta \in \partial \mathbb{D}} \{f(y \pm) : y \in M_0^\pm(QC) \setminus M_0^0(QC)\}$$

$$\bigcup \{tf(y+) + (1 - t)f(y-) : y \in M_0^0(QC), t \in [0, 1]\}.$$
Proof. By spectral permanence, (3.5), (3.6), \( p(0,0) = 0 \), Theorem 3.6 and (2.3),
\[
\sigma_e(T_f + p(C_\phi, C_\phi^*)) = \{x([T_f]) + p(x([C_\phi]), x([C_\phi])) : x \in M([T_\mathcal{C}(PQC(\zeta), \phi)])\}
\]
\[
= \{x([T_f]) : x \in M_0\} \bigcup \left\{ f(\zeta) + p(\lambda, \bar{\lambda}) : \lambda \in e^{-\alpha[0,\infty]} \right\}
\]
\[
= \sigma_e(T_f) \bigcup (f(\zeta) + \sigma_e(p(C_\phi, C_\phi^*))).
\] (3.7)

The essential norms equal the essential spectral radii in this case, while \( \|T_f\|_e = \|f\|_\infty \) for any \( L_\infty \) symbol [8, Chapter 7]. So the essential norm formula follows from (3.7).

Considering the fibers of \( M([T_\mathcal{C}(PQC(\zeta)]) \) over \( \partial \mathbb{D} \), one finds \( \sigma_e(T_f) \) via Theorem 2.3. \( \square \)

**Corollary 3.8.** For \( f \in PQC(\zeta) \) and a polynomial \( p \), \( \sigma_e(T_f + p(C_\phi, C_\phi^*)) \) is connected.

Proof. Subtracting a constant from \( p \) translates the essential spectrum. So we assume \( p(0,0) = 0 \). By a theorem of R. G. Douglas [8, Theorem 7.45], \( \sigma_e(T_f) \) is connected for any \( L_\infty \) symbol. In view of the homeomorphism \( z \mapsto |z| \) between \( e^{-\alpha[0,\infty]} \) and \([0,1] \), the second set in the union in (3.7) is also connected. Since the two connected sets intersect (both containing \( f(\zeta) \)), the union \( \sigma_e(T_f + p(C_\phi, C_\phi^*)) \) is connected. \( \square \)

Note in passing that there are harmonic Toeplitz operators on the Bergman space with disconnected essential spectra [17].

The next result states that the index of a Fredholm operator in \( T_\mathcal{C}(PQC(\zeta), \phi) \) is the same as that of its Toeplitz component, while the latter index is determined in [15, Theorems 2, 4].

**Theorem 3.9.** If \( T \in T_\mathcal{C}(PQC(\zeta)) \) and a bivariate polynomial \( p \) with \( p(0,0) = 0 \) are such that \( T + p(C_\phi, C_\phi^*) \) is Fredholm, then \( \text{ind}(T + p(C_\phi, C_\phi^*)) = \text{ind}(T) \).

Proof. Consider the homeomorphism \( \tau : z \in e^{-\alpha[0,\infty]} \mapsto |z| \in [0,1] \). To each \( t \in [0,1] \), define \( k_t : r \in [0,1] \mapsto (1-t)r \in [0,1] \) and put
\[
h_t = \tau^{-1} \circ k_t \circ \tau \in C_0(e^{-\alpha[0,\infty]}).
\]
Then \( h_0(z) = z \), \( h_1(z) \equiv 0 \), and \( h_t \) depends continuously on \( t \) for \( k_t \) does so. Put
\[
a_t = [T] + p(h_t([C_\phi]), \overline{h_t([C_\phi])}),
\]
and one has a continuous path \( t \mapsto a_t \) in \( [T_\mathcal{C}(PQC(\zeta), \phi)] \) joining \( a_0 = [T] + p([C_\phi], [C_\phi^*]) \) to \( a_1 = [T] \). Since \( h_t \) for each \( t \in [0,1] \) maps \( e^{-\alpha[0,\infty]} \) into itself, a deduction similar to the proof of Theorem 3.7 reveals that the range of the Gelfand transform \( \hat{a}_t \) is contained in that of \( \hat{a}_0 \), the latter being disjoint from the origin by hypothesis. Thus \( [T + p(C_\phi, C_\phi^*)] \) and \( [T] \) belong to the same component of \( [T_\mathcal{C}(PQC(\zeta), \phi)]^{-1} \), a fortiori to the same component of the group of invertible elements in the Calkin algebra. The index equality follows at once. \( \square \)
The group of invertible elements in a commutative Banach algebra $A$ is denoted by $A^{-1}$ with the subgroup $A_0^{-1} = eA$ being its unital component. The cosets in the abstract index group $A^{-1}/A_0^{-1}$ are precisely the components of $A^{-1}$. Recall the closed subset $M_0 \subset M([\mathcal{T}(\text{PQC}(\zeta), \phi)])$ defined in (3.6). The first Čech cohomology group of a compact Hausdorff space $\Omega$ is written $H^1(\Omega)$.

**Proposition 3.10.** $H^1(M([\mathcal{T}(\text{PQC}(\zeta), \phi)])) \cong H^1(M_0)$.

**Proof.** Consider the homomorphism from the abstract index group of $[\mathcal{T}(\text{PQC}(\zeta))]$ into that of $[\mathcal{T}(\text{PQC}(\zeta), \phi)]$, mapping every component of $[\mathcal{T}(\text{PQC}(\zeta))]^{-1}$ to that of $[\mathcal{T}(\text{PQC}(\zeta), \phi)]^{-1}$ which contains the former. The map is surjective. For, by Lemma 3.4 let $[T] + f([C_\phi]) \in [\mathcal{T}(\text{PQC}(\zeta), \phi)]^{-1}$ where $T \in \mathcal{T}(\text{PQC}(\zeta))$ and $f \in C_0(e^{-\alpha[0, \infty)}) = zC(e^{-\alpha[0, \infty)}) = \{zp(z, \bar{z}) : p \text{ a bivariate polynomial}\}$ in uniform closures over $e^{-\alpha[0, \infty)}$. Then the component of $[\mathcal{T}(\text{PQC}(\zeta), \phi)]^{-1}$ containing $[T] + f([C_\phi])$ contains $[T] + p([C_\phi], [C_\phi^*])$ for some $p$ with $p(0, 0) = 0$, and in turn contains $[T]$ together with its component of $[\mathcal{T}(\text{PQC}(\zeta))]^{-1}$, by the proof of Theorem 3.9. This verifies surjectivity.

To see that the map is also injective, suppose $[T_1], [T_2] \in [\mathcal{T}(\text{PQC}(\zeta))]^{-1}$ share a common component of $[\mathcal{T}(\text{PQC}(\zeta), \phi)]^{-1}$. That is $[T_1] = [T_2]e^a$ for some $a \in [\mathcal{T}(\text{PQC}(\zeta), \phi)]$. It follows that the Gelfand transform $\hat{a}$ assumes discrete values on the fiber over $\{\zeta\}$ because $e^a$ is constant there. Since this fiber is homeomorphic to $e^{-\alpha[0, \infty]}$ (Theorem 3.6) and to $[0, 1]$, the range of the continuous function $\hat{a}$ on the fiber must be connected hence constant. Since all other fibers of $M([\mathcal{T}(\text{PQC}(\zeta), \phi)])$ over $M([\mathcal{T}(\text{PQC}(\zeta))])$ are singletons, we assert $a \in [\mathcal{T}(\text{PQC}(\zeta))]$. So, $[T_1], [T_2]$ share a common component of $[\mathcal{T}(\text{PQC}(\zeta))]^{-1}$, and the map is injective.

Now that the two abstract index groups are isomorphic, so are the groups $H^1(M([\mathcal{T}(\text{PQC}(\zeta), \phi)]))$ and $H^1(M([\mathcal{T}(\text{PQC}(\zeta))]))$. But the latter group is isomorphic to $H^1(M_0)$ in view of the natural homeomorphism from $M_0$ onto $M([\mathcal{T}(\text{PQC}(\zeta))])$ given by restriction. This completes the proof.

4. The non-parabolic case

Assume in this section $\phi$ is non-parabolic, that is, $\phi'(\zeta) \neq 1$ at the fixed boundary point $\zeta$. The C*-algebra $[\mathcal{T}(\text{PQC}(\zeta), \phi)]$ is not commutative in this case. Let $\mathcal{A}$ be the (non-self-adjoint) norm-closed algebra generated by $\{C_\phi, T_f : f \in \text{PQC}(\zeta)\}$. Since $\mathcal{A} \supset \mathcal{T}(C) \supset \mathcal{K}$, the quotient algebra $[\mathcal{A}]$ equals the closed subalgebra of the Calkin algebra generated by $\{(C_\phi), [T_f] : f \in \text{PQC}(\zeta)\}$, a commutative Banach algebra containing the C*-subalgebra $[\mathcal{T}(\text{PQC}(\zeta))]$. The fibers of $M([\mathcal{A}])$ over $M([\mathcal{T}(\text{PQC}(\zeta))])$ will be similarly described as in Theorem 3.1 and 3.6 for the C*-algebra. Although the essential spectrum $\sigma_e(C_\phi)$ lacks an explicit form in this case, what we need is only its polynomial-convex hull.

**Lemma 4.1.** $\text{hull}(\sigma_e(C_\phi)) = \{|z| \leq \sqrt{s}\}$. 
Proof. It is known that $\phi'(\zeta) > 0$ (cf. [12], p. 744). If $\phi'(\zeta) < 1$ at the Denjoy-Wolff point $\zeta$ [7, p. 59], then [7] Theorem 7.26 applies and asserts that $\sigma(C_\phi) = \{|z| \leq \sqrt{s}\}$, $s = 1/\phi'(\zeta)$, and [4] Theorem 37.8 in turn gives
\[\{|z| = \sqrt{s}\} \subset \sigma_e(C_\phi) \subset \{|z| \leq \sqrt{s}\}. \tag{4.1}\]

If otherwise $\phi'(\zeta) > 1$, then $\psi'(\zeta) = 1/\phi'(\zeta) < 1$ for the Krein adjoint $\psi$ of $\phi$ [11, Prop. 3.4], and the derivation above applies to $\psi$ at its Denjoy-Wolff point $\zeta$ to give
\[\{|z| = 1/\sqrt{s}\} \subset \sigma_e(C_\psi) \subset \{|z| \leq 1/\sqrt{s}\}\]
Since $\sigma_e(C_\phi^*) = s\sigma_e(C_\psi)$ due to $[C_\phi^*] = s[C_\psi]$, multiplying these inclusions by $s$ and taking complex conjugates yield (4.1) again.

Since the circle $\{|z| = \sqrt{s}\}$ disconnects the plane, it follows from (4.1) that the unbounded connected component of the complement of $\sigma_e(C_\phi)$ is the complement of the disc $\{|z| \leq \sqrt{s}\}$. The proof is complete by the well-known characterization of the hull. \square

Remark 4.2. To (4.1) one can add $0 \in \sigma_e(C_\phi)$. For, the injective but non-invertible [7] $C_\phi$ of a linear fractional non-automorphism has dense [6] but not closed range, hence not a Fredholm operator. Incidentally, this also follows from Theorem 4.9 with $T = 0$, $p(z) = z$. Also, it follows from the lemma and the essential norm formula in terms of Aleksandrov-Clark measures [3] that
\[\|[C_\phi]\| = \sqrt{s} = \rho([C_\phi])\]
for the non-normal $[C_\phi]$. It would be interesting to know if this equality of essential norm and essential spectral radius extends to polynomials of $C_\phi$.

The counterpart to Theorem 3.1 is true for $[A]$ with an identical proof.

Theorem 4.3. For every $\xi \neq \langle \zeta \rangle \in M([T(PQC(\zeta))])$, the fiber $M_\xi([A])$ consists of a single functional vanishing at $[C_\phi]$.

To prove the counterpart to Theorem 3.6, we need some variants of the three lemmas. We shall only outline their proof, if not omitting it. Let $P_0([C_\phi])$ be the norm-closure of
\[\{[C_\phi]p([C_\phi]) : p \text{ is a polynomial}\}.

Lemma 4.4. For every $T \in T(PQC(\zeta))$ and $b \in P_0([C_\phi])$,
\[\[T\]b = b[\langle \xi \rangle] = \langle \xi \rangle(\[T\])b.\]

Lemma 4.5. There exists $y \in M(\langle \zeta \rangle([A]))$ such that $y([C_\phi]) = 0$.

Proof. Such $y$ arises as a cluster point as before, using Theorem 4.3 instead. \square

Lemma 4.6. For every $T \in T(PQC(\zeta))$ and $b \in P_0([C_\phi])$,
\[\|[T] + b\| \geq \|[T]\|. \tag{4.2}\]

Consequently, one has the Banach space direct-sum decomposition
\[A = [T(PQC(\zeta))] \bigoplus P_0([C_\phi]).\]
Proof. Relative to the commutative C*-algebra \( \mathcal{T}(PQC(\zeta)) \) one has
\[
\|T\| = \max\{|\xi(T)| : \xi \in M(\mathcal{T}(PQC(\zeta)))\},
\]
while relative to the commutative Banach algebra \( \mathcal{A} \) one has
\[
\|T + b\| \geq \max\{|y(T)| + y(b) : y \in M(\mathcal{A})\}.
\]
Partitioning \( M(\mathcal{A}) \) into fibers over \( M(\mathcal{T}(PQC(\zeta))) \) and using Theorem 4.3 and Lemma 4.5, one verifies that the second maximum is no less than the first. Note that \( y([C_\phi]) = 0 \) implies \( y(b) = 0 \) by multiplicativity and continuity. The direct-sum decomposition follows from (4.2) as usual, noting Lemma 4.4 and the fact that \( P_0([C_\phi]) \) is norm-closed. \( \square \)

**Theorem 4.7.** The fiber \( M(\zeta)(\mathcal{A}) \) is homeomorphic to \( \{ |z| \leq \sqrt{s} \} \) via the map
\[
y \in M(\zeta)(\mathcal{A}) \mapsto y([C_\phi]) \in \{ |z| \leq \sqrt{s} \}.
\]
Proof. Let \( A([C_\phi]) \) be the non-self-adjoint closed Calkin subalgebra singly generated by \([C_\phi] \). The injective continuous map has its range in
\[
\sigma([C_\phi], \mathcal{A}) \subset \sigma([C_\phi], A([C_\phi])) = \hull(\sigma_e(C_\phi)) = \{ |z| \leq \sqrt{s} \}
\]
by (2.4) and Lemma 4.1. Conversely, every \( \lambda \in \sigma([C_\phi], A([C_\phi])) = \{ |z| \leq \sqrt{s} \} \) equals \( m([C_\phi]) \) for some multiplicative linear functional \( m \) on the commutative Banach algebra \( A([C_\phi]) \). Letting \( m' := m|P_0([C_\phi]) \), one directly verifies that the direct-sum linear functional \( \langle \zeta \rangle \oplus m' \) on \( \mathcal{A} = [\mathcal{T}(PQC(\zeta))] \oplus P_0([C_\phi]) \) is multiplicative using Lemma 4.4. Thus,
\[
y := \langle \zeta \rangle \oplus m' \in M(\zeta)(\mathcal{A}) \text{ with } y([C_\phi]) = m([C_\phi]) = \lambda,
\]
establishing surjectivity and completing the proof. \( \square \)

Letting \( U \) be the Calkin algebra and \( B = [\mathcal{A}] \) in (2.5), one has for every \( T \in \mathcal{A} \) the inclusions
\[
\{ y(T) : y \in \partial[\mathcal{A}] \} \subset \sigma_e(T) \subset \{ y(T) : y \in M(\mathcal{A}) \}. \tag{4.3}
\]
The Shilov boundary \( \partial[\mathcal{A}] \) turns out to be a fairly large subset of \( M([\mathcal{A}]) \) and can be explicitly identified. Denote by \( (\langle \zeta \rangle, \lambda) \) the functional \( y \in M(\zeta)(\mathcal{A}) \) with \( y([C_\phi]) = \lambda \).

**Theorem 4.8.** With \( \xi \) ranging over \( M([\mathcal{T}(PQC(\zeta))]) \),
\[
\partial[\mathcal{A}] = \bigsqcup_{\xi \neq \langle \zeta \rangle} M_\xi([\mathcal{A}]) \bigsqcup \{ (\langle \zeta \rangle, \lambda) : |\lambda| = 0, \sqrt{s} \}.
\]
Proof. Throughout the proof denote by \( F \) the union on the right side. \( F \) is a closed subset of \( M([\mathcal{A}]) \) because it is the pre-image of a closed set under a continuous function:
\[
F = \{ y \in M([\mathcal{A}]) : |y([C_\phi])| = 0, \sqrt{s} \}.
\]

We shall first show \( F \) is a boundary for \([\mathcal{A}]\). Let \( T \in \mathcal{T}(PQC(\zeta)) \) and let \( p \) be a polynomial. Write \( a = [T] + [C_\phi]p([C_\phi]) \) with Gelfand transform \( \hat{a} \) on \( M([\mathcal{A}]) \). By the maximum modulus principle,
\[
\max_{|\lambda| \leq \sqrt{s}} |\langle \zeta \rangle([T]) + \lambda p(\lambda)| = \max_{|\lambda| = 0 \vee \sqrt{s}} |\langle \zeta \rangle([T]) + \lambda p(\lambda)|. \tag{4.4}
\]
Partitioning $M([A])$ into fibers over $M([\mathcal{T}(PQC(\zeta))])$, we have by their descriptions in Theorem 4.3, 4.7 that

$$\|\hat{a}\|_\infty = \sup_{\xi \neq \langle \zeta \rangle} |\xi([T])| \vee \max_{|\lambda| \leq \sqrt{s}} |\langle \zeta \rangle([T]) + \lambda p(\lambda)|$$

$$\|\hat{a}|F\|_\infty = \sup_{\xi \neq \langle \zeta \rangle} |\xi([T])| \vee \max_{|\lambda| = 0, \sqrt{s}} |\langle \zeta \rangle([T]) + \lambda p(\lambda)|$$

and obtain by (4.4) $\|\hat{a}\|_\infty = \|\hat{a}|F\|_\infty$. This norm equality on $M([A])$ extends to every element of $[A]$ due to the decomposition in Lemma 4.6 and density of $[C_\phi]p([C_\phi])$ in $P_0([C_\phi])$. That is, the closed subset $F$ is a boundary for $[A]$, thus

$$F \supset \partial[A].$$

Conversely, consider the restriction map $r : M([A]) \to M([\mathcal{T}(PQC(\zeta))]).$ Since for the C*-algebra

$$\partial[\mathcal{T}(PQC(\zeta))] = M([\mathcal{T}(PQC(\zeta))]),$$

Proposition 2.5 implies that the singleton fiber (Theorem 4.3) $M_\zeta([A])$ lies on $\partial[A]$ for every $\xi \neq \langle \zeta \rangle$. Next, using a cluster point argument as before, $(\langle \zeta \rangle, 0)$ lies in the closure of the union of these fibers, and hence also in the closed set $\partial[A]$. It remains only to show for an arbitrary $\lambda$, $|\lambda| = \sqrt{s}$, that $(\langle \zeta \rangle, \lambda) \in \partial[A]$. To this end let

$$b := \lambda[I] + [C_\phi] \in [A].$$

Since $y(b) \equiv \lambda$ on $\bigcup_{\xi \neq \langle \zeta \rangle} M_\xi([A])$ and $(\langle \zeta \rangle, \mu)(b) = \lambda + \mu$, $|\mu| \leq \sqrt{s}$, $\hat{b}$ peaks at $(\langle \zeta \rangle, \lambda)$ with $\|\hat{b}\|_\infty = 2\sqrt{s}$. This is because for $|\lambda| = \sqrt{s}$ and $|\mu| \leq \sqrt{s}$,

$$|\lambda + \mu| = 2\sqrt{s} \iff \mu = \lambda.$$

The peak point $(\langle \zeta \rangle, \lambda)$ for $[A]$ must lie on $\partial[A]$, as required. \qed

These results together allow for a reasonable estimate of certain essential spectra, which is the main outcome of this section. By Theorem 4.3,

$$N_0 := \bigcup_{\lambda \neq \zeta \in \partial D} M_\xi([A]) \bigcup \{\zeta, 0\} = \{y \in M([A]) : y([C_\phi]) = 0\}. \quad (4.5)$$

**Theorem 4.9.** For $T \in \mathcal{T}(PQC(\zeta))$ and a polynomial $p$ with $p(0) = 0$,

$$\sigma_e(T) \bigcup \{\langle \zeta \rangle([T]) + p(\lambda) : |\lambda| = \sqrt{s}\} \subset \sigma_e(T + p(C_\phi))$$

$$\subset \sigma_e(T) \bigcup \{\langle \zeta \rangle([T]) + p(\lambda) : |\lambda| \leq \sqrt{s}\}.$$

**Proof.** By (4.5), $p(0) = 0$, and Theorem 4.7

$$\{y([T + p(C_\phi)]) : y \in M([A])\} = \{y([T]) + p(y([C_\phi])) : y \in M([A])\}$$

$$= \{y([T]) : y \in N_0\} \bigcup \{\langle \zeta \rangle([T]) + p(\lambda) : |\lambda| \leq \sqrt{s}\}$$

$$= \sigma_e(T) \bigcup \{\langle \zeta \rangle([T]) + p(\lambda) : |\lambda| \leq \sqrt{s}\},$$
the last step due to spectral permanence in $[\mathcal{T}(PQC(\zeta))]$. A similar deduction using Theorem 4.8 yields
\[\{y([T + p(C_\phi)]): y \in \partial[A]\} = \sigma_e(T) \bigcup \{\langle \zeta \rangle([T]) + p(\lambda): |\lambda| = \sqrt{s}\}.\]
These together with (4.3) for $T + p(C_\phi) \in \mathcal{A}$ give the desired inclusions. □

In particular, the set difference between the bounds for $\sigma_e(T + p(C_\phi))$ is no larger than a translate of \{\(p(\lambda): 0 < |\lambda| < \sqrt{s}\}\}. In comparison to the bounds (4.1) for $\sigma_e(C_\phi)$ and Remark 4.2, the perturbation by an operator from the Toeplitz subalgebra $T(PQC(\zeta))$ does not increase the difference.

The following result is a weaker counterpart to Theorem 3.9, for the condition is stronger than $T + p(C_\phi)$ being Fredholm. The proof is similar and omitted, noting that the disc $\{|\lambda| \leq \sqrt{s}\}$ linearly contracts to the origin just like $[0,1]$.

**Theorem 4.10.** If $T \in T(PQC(\zeta))$ and a polynomial $p$ with $p(0) = 0$ are such that $0 \notin \sigma_e(T) \bigcup \{\langle \zeta \rangle([T]) + p(\lambda): |\lambda| \leq \sqrt{s}\}$, then $\text{ind}(T + p(C_\phi)) = \text{ind}(T)$.

We close this section by mentioning that the counterpart to Proposition 3.10, with the subset $N_0 \subset M([A])$ in (4.5) replacing $M_0$, would hold if only the algebra singly generated by $[C_\phi]$ were semi-simple (see the end of Remark 4.2).

**References**

Yi Yan
Department of Mathematics, University of Kansas, Lawrence, KS 66045, USA
e-mail: yiyan@ku.edu
